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Second-Order Formalism for 3D Spin-3 Gravity

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Abstract

A second-order formalism for the theory of 3D spin-3 gravity is considered. Such a formalism is obtained by solving the torsion-free condition for the spin connection ω_μ^a, and substituting the result into the action integral. In the first-order formalism of the spin-3 gravity defined in terms of $SL(3,R) \times SL(3,R)$ Chern-Simons (CS) theory, however, the generalized torsion-free condition cannot be easily solved for the spin connection, because the vielbein e_μ^a itself is not invertible. To circumvent this problem, extra vielbein-like fields $e_{\mu\nu}^a$ are introduced as a functional of e_μ^a. New set of affine-like connections $\Gamma_{\mu M}^N$ are defined in terms of the metric-like fields, and a generalization of the Riemann curvature tensor is also presented. In terms of this generalized Riemann tensor the action integral in the second-order formalism is expressed. The transformation rules of the metric and the spin-3 gauge field under the generalized diffeomorphims are obtained explicitly. As in Einstein gravity, the new affine-like connections are related to the spin connection by a certain gauge transformation, and a gravitational CS term expressed in terms of the new connections is also presented.

1 Introduction

Gravity theories coupled to massless higher spin fields have been studied extensively in recent years.[1][2][3][4] [5][6][7][8][9][10][11] [12][13][14][15][16][17][18][19] This is due to the conjectured holographic relation between these theories in AdS_4 and the O(N) vector model in 3D.[20] These higher-spin theories contain an infinite number of fields. [21][22][23][24]

Three dimensional higher-spin theories[25] are simpler to work with than those in higher dimensions, because higher-spin fields can be truncated to only those with spin $s \leq N$ and the theory can be defined in terms of the Chern-Simons action.[2] By using the $SL(3,R) \times SL(3,R)$ invariant Chern-Simons (CS) theory, the black hole solution with spin-3 charge in spin-3 gravity was obtained and studied.[6][10][11][17]

The CS theory for gravity is very efficient for obtaining solutions. The condition of flat connections is easy to implement. The asymptotic behavior of the solution was found to be not necessarily $\mathcal{O}(r^2)$. One needs to perform spin-3 gauge transformations to transform the black hole solution into a form with a manifest event horizon.[6][10] The geometry of 3D higher spin gravity, which must be a generalization of the Riemannian geometry, is not well understood. It is difficult to understand this within the CS approach. Therefore, it is necessary to understand the geometry of higher-spin gravity in more details by a different approach. Additionally, general integral formulae for the spin-3 charge and the entropy are not yet derived. The explicit action integrals for matter fields coupled to higher-spin gravity is also not known.

In the spin-2 gravity theory there exist first-order and second-order formalisms. In the first-order formalism, a spin connection and a vielbein field are introduced, and the action integral contains only the first-order derivatives of the fields. In the second-order formalism, the spin connection is eliminated by solving the torsion-free condition, and the solution is substituted into the action integral. Then the action integral becomes quadratic in the derivatives. Both formalisms are equivalent. The CS formulation of the spin-3 gravity is the first-order one. It is expected that a second-order formalism also exists for the spin-3 gravity. In order to tackle this problem it is necessary to reformulate the theory of higher-spin gravity as a geometrical theory in terms of the metric-like fields. For this purpose one needs to introduce affine-like connections, covariant derivatives and the curvature tensors into the spin-3 theory. Since it is not clear what these concepts mean, we need to study these matters in a heuristic way.

Apparently, this problem is difficult to solve, because in $SL(N,R) \times SL(N,R)$ CS theory (with $N \geq 3$), the dimension of the algebra $sl(N,R)$ is larger than that of spacetime. The

vielbein e^a_μ is not a square matrix and does not have an inverse.[1] For example, in the $N=3$ case which corresponds to spin-3 gravity, a runs from 1 to 8 and μ from 0 to 2. In this paper, to compensate this gap of the numbers of components, auxiliary vielbein fields $e^a_{\mu\nu}$ will be introduced. With the traceless and symmetry conditions $g^{\mu\nu}\, e^a_{\mu\nu} = 0$, $e^a_{\mu\nu} = e^a_{\nu\mu}$, the entire vielbein field becomes an 8×8 matrix. So if this generalized vielbein field is non-degenerate, an inverse vielbein exists and the torsion-free condition can be solved.

Now, many concepts and geometrical quantites can be introduced in the spin-3 gravity in parallel with the Einstein gravity. The purpose of this paper is to pursue this possibility. For example, two connections, $\Gamma^\lambda_{\mu\nu}$ and $\Gamma^{\lambda\rho}_{\mu\nu}$, are obtained by generalization of the vielbein postulate, $\partial_\mu\, e^a_\nu + f^a{}_{bc}\, \omega^b_\mu\, e^c = \Gamma^\lambda_{\mu\nu}\, e^a_\lambda + \frac{1}{2}\, \Gamma^{\lambda\rho}_{\mu\nu}\, e^a_{\lambda\rho}$, instead of the Christoffel symbol $\hat{\Gamma}^\lambda_{\mu\nu}$ in the Einstein gravity. These connections are expected to be used to describe the geometry of spin-3 gravity. By using these connections appropriate covariant derivatives ∇_μ can be introduced in such a way that the full covariant derivatives D_μ of the vielbeins, $D_\mu\, e^a_\nu = \nabla_\mu\, e^a_\nu + f^a{}_{bc}\, \omega^b_\mu\, e^c_\nu$ and $D_\mu\, e^a_{\nu\lambda} = \nabla_\mu\, e^a_{\nu\lambda} + f^a{}_{bc}\, \omega^b_\mu\, e^c_{\nu\lambda}$, vanish. In the definition of $\nabla_\mu\, e^a_{\nu\lambda}$, two more connections, $\Gamma^\rho_{\mu,(\nu\lambda)}$ and $\Gamma^{\rho\sigma}_{\mu,(\nu\lambda)}$, are also defined. It then turns out convenient to combine e^a_μ and $e^a_{\mu\nu}$ into a single vielbein field e^a_M, where M takes two kinds of indices, $M=\mu$ and $M=(\mu\nu)$. Here (μ,ν) denotes a traceless, symmetric pair of base-space indices. Similarly, the four affine-like connections can be combined as $\Gamma^N_{\mu M}$. The metric tensor can also be generalized: defining $G_{MN} = e^a_M\, e_{aN}$, which is generalization of the metric tensor $g_{\mu\nu} = e^a_\mu\, e_{a\nu}$, we find that G_{MN} is compatible with the covariant derivative ∇_μ.

The covariant-constancy conditions of e^a_μ and $e^a_{\mu\nu}$ are solved to yield the spin connection $\omega^a_\mu(e)$ as a functional of the vielbeins.

$$\omega^a_{\mu\ c} \equiv f^a{}_{bc}\, \omega^b_\mu = -E^\nu_c\, \nabla_\mu\, e^a_\nu - \frac{1}{2}\, E^{\nu\lambda}_c\, \nabla_\mu\, e^a_{\nu\lambda} \tag{1.1}$$

Here E^μ_a and $E^{\mu\nu}_a$ are inverse vierbeins. By substituting this into the CS action, the second-order action is obtained. Furthermore, from the connections, $\Gamma^N_{\mu M}$, generalized curvature tensors, $R^M{}_{N\lambda\rho}$, can be defined and the action integral can be expressed in terms of these curvature tensors.

$$S_{\text{2nd order}} = \frac{k}{12\pi} \int d^3x\, \left\{ -\epsilon^{\mu\nu\lambda}\, (f^a{}_{bc}\, e^c_\mu\, e^b_M\, E^N_a)\, R^M{}_{N\nu\lambda} + 4\, \epsilon^{\mu\nu\lambda}\, f^a{}_{bc}\, e^a_\mu\, e^b_\nu\, e^c_\lambda \right\} \tag{1.2}$$

As in Einstein gravity, the connections $\Gamma^N_{\mu M}$ and the spin connection ω^a_μ turn out to be related by a gauge transformation. By using this fact the gravitational CS term $S_{\text{GCS}}(\Gamma)$ can be explicitly expressed in terms of the connections (Γ's) and the topologically massive

[1] In [2] the torsion-free condition in spin-s gravity is solved to first order of expansion around a spin-2 background. In this paper the torsion-free condition will be solved explicitly without using perturbative expansions.

spin-3 gravity theory is defined.

$$
\begin{aligned}
S_{\rm GCS}^{\text{spin-3}}(\Gamma) \;=\; & \frac{k}{8\pi\mu}\int d^3x\,\epsilon^{\mu\nu\lambda}\left(\Gamma^{\rho}_{\mu\sigma}\partial_\nu\,\Gamma^{\sigma}_{\lambda\rho}+\frac{1}{2}\,\Gamma^{\rho\sigma}_{\mu\kappa}\,\partial_\nu\,\Gamma^{\kappa}_{\lambda,(\rho\sigma)}+\frac{1}{2}\,\Gamma^{\kappa}_{(\rho\sigma)}\,\partial_\nu\,\Gamma^{\rho\sigma}_{\lambda\kappa}\right.\\
& +\frac{1}{4}\,\Gamma^{\rho\sigma}_{\mu,(\kappa\tau)}\,\partial_\nu\,\Gamma^{\kappa\tau}_{\lambda,(\rho\sigma)}+\frac{2}{3}\,\Gamma^{\rho}_{\mu\sigma}\,\Gamma^{\sigma}_{\nu\kappa}\,\Gamma^{\kappa}_{\lambda\rho}+\Gamma^{\rho\tau}_{\mu,\sigma}\,\Gamma^{\sigma}_{\nu\kappa}\,\Gamma^{\kappa}_{\lambda,(\rho\tau)}\\
& \left.+\frac{1}{2}\,\Gamma^{\rho\tau}_{\mu,(\sigma\eta)}\,\Gamma^{\sigma\eta}_{\nu\kappa}\,\Gamma^{\kappa}_{\lambda,(\rho\tau)}+\frac{1}{12}\,\Gamma^{\rho\tau}_{\mu,(\sigma\eta)}\,\Gamma^{\sigma\eta}_{\nu(\kappa\alpha)}\,\Gamma^{\kappa\alpha}_{\lambda,(\rho\tau)}\right). \qquad (1.3)
\end{aligned}
$$

From the gauge transformations of the CS theory the generalized diffeomorphism of the metric field $g_{\mu\nu}$ (2.10) and the spin-3 gauge field $\phi_{\mu\nu\lambda}$ (2.11) can be defined and computed. These generalized diffeomorphisms are the ordinary 3D diffeomorphism and the spin-3 gauge transformation. It will be checked that these fields appropriately transform as spin-two and spin-three fields, respectively, under the ordinary diffeomorphism. New transformation rules of these fields under the spin-3 gauge transformation will also be obtained explicitly. The results are compactly written as

$$
\begin{aligned}
\delta\,g_{\mu\nu} \;=\; & \nabla_\mu\,\xi_\nu+\nabla_\nu\,\xi_\mu, \qquad (1.4)\\
\delta\,\phi_{\mu\nu\lambda} \;=\; & \nabla_\mu\left(\xi_{(\nu\lambda)}+\frac{1}{3}\,\rho^a\,\Lambda_a\,g_{\nu\lambda}\right)+\nabla_\nu\left(\xi_{(\lambda\mu)}+\frac{1}{3}\,\rho^a\,\Lambda_a\,g_{\lambda\mu}\right)\\
& +\nabla_\lambda\left(\xi_{(\mu\nu)}+\frac{1}{3}\,\rho^a\,\Lambda_a\,g_{\mu\nu}\right), \qquad (1.5)
\end{aligned}
$$

where ξ_μ and $\xi_{(\mu\nu)}$ are parameters of the generalized diffeomorphisms, and related to a gauge function Λ^a (5.2), (5.3). ρ^a is defined by $\rho^a=\frac{1}{2}\,d^a{}_{bc}\,e^b_\lambda\,e^c_\rho\,g^{\lambda\rho}$.

This paper is organized as follows. Sec.2 is a brief review of 3D spin-3 gravity as a CS theory. In sec.3 a method for an extension of the vielbein field is explained. Then two connections, $\Gamma^{\lambda}_{\mu\nu}$, $\Gamma^{\lambda\rho}_{\mu\nu}$, analogous to the Christoffel symbol are introduced. In sec.4 the spin connection is determined as a functional of the vielbein. The covariant derivatives of the vielbeins are defined and two more connections, $\Gamma^{\rho}_{\mu,(\nu\lambda)}$, $\Gamma^{\rho\sigma}_{\mu,(\nu\lambda)}$, are introduced. We obtain an invariant action integral for the spin-3 gravity in the second-order formalism. We point out that the indices of tensors are shown to have a novel pairing structure, μ and $(\mu\nu)$. In sec.5 the transformation rules of the the metric field $g_{\mu\nu}$ and the spin-3 gauge field $\phi_{\mu\nu\lambda}$ under the generalized diffeomorphism will be computed. In sec.6 curvature tensors for the spin-3 geometry is defined. In sec.7 the spin-3 gravity version of the gravitational CS term is derived. Finally, sec.8 is reserved for summary and discussions. There are appendices A-E. Appendix A summarizes the $sl(3,R)$ formulae. In appendix B it will be shown that for the AdS$_3$ background the vielbein system e^a_μ, $e^a_{\mu\nu}$ is actually non-degenerate. The inverse vielbeins and other quantities are obtained. Killing vectors ξ_μ and tensors $\xi_{(\mu\nu)}$ for AdS$_3$ are also presented. In appendix C it is pointed out that in addition to the metric $g_{\mu\nu}$ and the spin-3 gauge field $\phi_{\mu\nu\lambda}$, extra gauge fields such as $g_{(\mu\nu)(\mu\nu)}$, $g_{(\mu\nu)(\lambda\rho)\sigma}$, $g_{(\mu\nu)(\lambda\rho)(\sigma\kappa)}$ with

the number of indices up to 6 must be introduced. In appendix D the complicated part $S_{\mu\nu,\lambda\rho}$ of the connection $\Gamma^{\lambda\rho}_{\mu\nu}$ is obtained explicitly. In appendix E a metric tensor G_{MN} for 8D space, which corresponds to the two types of indices $M = \mu, (\mu\nu)$ and composed of the metric, the spin-3 gauge field and another gauge field, is introduced and some formulae for this metric tensor are derived.

Note Added: While this work was being completed, we found that a work[35] appeared in the arXiv, which attempts to formulate 3D spin-3 gauge theory in terms of the metric-like fields, by means of the perturbation in powers of the spin-3 gauge field $\phi_{\mu\nu\lambda}$ up to $\mathcal{O}(\phi_{\mu\nu\lambda})^2$. The action integral and various transformation rules of metric-like fields obtained in the present paper are not based on perturbations in $\phi_{\mu\nu\lambda}$. Further, we also generalize the geometrical notions of Einstein gravity, such as the connections and the curvature tensors, to the spin-3 geometry.

2 3D spin-3 gravity as Chern-Simons theory

Let us start by briefly reviewing the 3D spin-3 gravity defined in terms of the Chern-Simons theory.

The 3D spin-3 gravity with a negative cosmological constant is defined by the $SL(3,R) \times SL(3,R)$ Chern-Simons (CS) actions.[26][27]

$$S_{CS} = \frac{k}{4\pi} \int \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) - \frac{k}{4\pi} \int \mathrm{tr}(\bar{A} \wedge d\bar{A} + \frac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A}) \qquad (2.1)$$

Here A and $\bar{A}$ are gauge-field one-forms and live in the fundamental representation of $sl(3,R)$. The constant k is the level of the CS actions and is related to the three dimensional Newton constant G_N. The AdS length ℓ is given by $k = \ell/4G_N$. The above action is invariant under $SL(3,R) \times SL(3,R)$ gauge transformations up to boundary terms.

$$\begin{aligned} \delta A &= d\Lambda + [A, \Lambda], \\ \delta \bar{A} &= d\bar{\Lambda} + [\bar{A}, \bar{\Lambda}] \end{aligned} \qquad (2.2)$$

Here Λ is an $sl(3,R)$ matrix. These gauge fields A, $\bar{A}$ are related to the vielbein one-form $e = e_\mu \, dx^\mu$ and the spin connection one-form $\omega = \omega_\mu \, dx^\mu$ by the relations.

$$A = \omega + \frac{1}{\ell} e, \qquad \bar{A} = \omega - \frac{1}{\ell} e \qquad (2.3)$$

Here x^μ ($\mu = 0, 1, 2$) is the coordinate of 3D spacetime. The Greek indices $\mu, \nu, ..$ will be used for spacetime indices and the Roman letters $a, b, ..$ will be used for internal space (local frame) indices.

In what follows ℓ will be set to 1. The above action is then written in terms of e and ω.

$$S_{CS} = \frac{k}{\pi} \int \mathrm{tr}\ e \wedge \left(d\omega + \omega \wedge \omega + \frac{1}{3}\ e \wedge e \right) \tag{2.4}$$

The vielbein and the spin connection are expanded in terms of the $sl(3, R)$ generators t_a. (See appendix A) [2]

$$e = e^a\, t_a = e^a_\mu\, t_a\, dx^\mu, \qquad \omega = \omega^a\, t_a = \omega^a_\mu\, t_a\, dx^\mu \tag{2.5}$$

The gauge transformations on e and ω fall into two groups.

a. Local frame rotation (or, extended Lorentz transformation)

$$\delta e \ = \ [e,\, \Lambda_+], \tag{2.6}$$

$$\delta\omega \ = \ d\,\Lambda_+ + [\omega,\, \Lambda_+], \qquad \Lambda_+ \equiv \frac{1}{2}\,(\Lambda + \bar{\Lambda}) \tag{2.7}$$

These transformations are Lorentz transformations and extended rotations in local frame.

b. Generalized diffeomorphism (or, extended local translation)

$$\delta e \ = \ d\,\Lambda_- + [\omega,\, \Lambda_-], \tag{2.8}$$

$$\delta\omega \ = \ [e,\, \Lambda_-], \qquad \Lambda_- \equiv \frac{1}{2}\,(\Lambda - \bar{\Lambda}) \tag{2.9}$$

These transformations are the ordinary spacetime diffeomorphism and the spin-3 gauge transformations.

As usual, the metric tensor $g_{\mu\nu}$ of the spacetime is defined in terms of the vielbein $e_\mu = e^a_\mu\, t_a$.

$$g_{\mu\nu} = \frac{1}{2}\,\mathrm{tr}\ e_\mu\, e_\nu = h_{ab}\ e^a_\mu\ e^b_\nu = e^a_\mu\ e_{a\nu} \tag{2.10}$$

For the definition of the Killing metric h_{ab} for the $sl(3, R)$ algebra see appendix A. Throughout this paper $g_{\mu\nu}$ is assumed to be non-degenerate. It is also assumed that its signature is $(-, +, +)$. Its inverse is denoted as $g^{\mu\nu}$. Also in the literature[2], the spin-3 gauge field $\phi_{\mu\nu\lambda}$ is defined.

$$\phi_{\mu\nu\lambda} = \frac{1}{4}\ \mathrm{tr}\ \{e_\mu, e_\nu\}\ e_\lambda \tag{2.11}$$

Here $\{\ ,\ \}$ is an anti-commutator. This tensor and $g_{\mu\nu}$ are independent fields. The number of components of e^a_μ is 24 and there are 8 independent local frame transformations. So there

[2] In [2] a notation e^a_μ $(a = 1, 2, 3)$ is used for the dreibein and the extra field e^{ab}_μ $(a, b = 1, 2, 3)$ is called spin-3 gauge field. We will not adopt this notation.

must be 16 independent degrees of freedom for the metric-like fields. The metric $g_{\mu\nu}$ and the spin-3 gauge field $\phi_{\mu\nu\lambda}$ have 6 and 10 independent components, respectively. Thus it is expected that $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$ are sufficient. However, in appendix C it will be shown that extra tensor fields must be defined for describing spin-3 gravity. Although these extra fields may not be independent of the above fields, explicit formulae relating them are nested and complicated.[3]

3 Extension of the vielbein

The eqs of motion for the CS theory (2.4) are given by the conditions of flat connections,

$$F \equiv dA + A \wedge A = 0, \qquad \bar{F} \equiv d\bar{A} + \bar{A} \wedge \bar{A} = 0, \tag{3.1}$$

and in terms of e and ω these eqs are rewitten as a torsion-free condition

$$\mathcal{T} \equiv \frac{1}{2}(F - \bar{F}) = de + e \wedge \omega + \omega \wedge e = 0 \tag{3.2}$$

and an Einstein-like eq with negative cosmological constant.

$$\mathcal{R} \equiv \frac{1}{2}(F - \bar{F}) = d\omega + \omega \wedge \omega + e \wedge e = 0 \tag{3.3}$$

This eq describes more degrees of freedom than those of gravity.

In the ordinary (spin-2) 3D gravity described by $SL(2,R) \times SL(2,R)$ CS action, the condition (3.2) can be solved for ω, if the dreibein e^a_μ is invertible. Then this solution is substituted into (3.3) and the Einstein eq for Anti-de-Sitter (AdS) space is obtained. Similarly, to formulate a second-order theory of spin-3 gravity in terms of the metric-like fields, $g_{\mu\nu}$, $\phi_{\mu\nu\lambda}$, .., it is necessary to first solve the torsion-free condition and express the spin connection ω in terms of e and then, substitute the result into the action (2.4). The vielbein e^a_μ $(a = 1, \ldots, 8; \mu = 0, 1, 2)$, however, has a form of a rectanglular matrix and is non-invertible, even if $g_{\mu\nu}$ is non-degenerate.

3.1 New vielbein $e^a_{\mu\nu}$

To resolve this difficulty, 5 more basis vectors in the local frame must be introduced. Let us define the $SL(3,R)$ matrices

$$\hat{e}_{\mu\nu} = \frac{1}{2}\,\{e_\mu, e_\nu\} - \frac{2}{3}\,g_{\mu\nu}\,\boldsymbol{I}. \tag{3.4}$$

The second term on the right proportional to the identity matrix $\boldsymbol{I}$ is added to ensure tracelessness of $\hat{e}$ as a matrix of $sl(3,R)$. This is symmetric in the indices μ, ν and there

[3] A method for obtaining some of these relations is suggested at the end of appendix E.

are 6 independent components. To reduce the number of components by one the trace with respect to the indices μ, ν must be subtracted.

$$
\begin{aligned}
e_{\mu\nu} &= \hat{e}_{\mu\nu} - \frac{1}{3}\, g_{\mu\nu}\, \rho = \frac{1}{2}\, \{e_\mu, e_\nu\} - \frac{1}{3}\, g_{\mu\nu}\, \rho - \frac{2}{3}\, g_{\mu\nu}\, \boldsymbol{I}, && (3.5)\\
\rho &\equiv g^{\lambda\rho}\, \hat{e}_{\lambda\rho} && (3.6)
\end{aligned}
$$

The matrix ρ is the trace part of $\hat{e}_{\mu\nu}$, and $e_{\mu\nu}$ satisfies $g^{\mu\nu}\, e_{\mu\nu} = 0$.[4] Although these additional matrices $e_{\mu\nu}$ functionally depend on e_μ, 8D vectors $e^a_{\mu\nu}$ $(a = 1, \ldots, 8)$ defined by $e_{\mu\nu} = e^a_{\mu\nu}\, t_a$ are assumed to span the 8D space together with e^a_μ. In appendix B the case of AdS$_3$ vacuum is examined and it is shown that this is the case. It can also be shown that extented vielbeins $(e^a_\mu,\, e^a_{\mu\nu})$ for the BTZ black hole embedded in the spin-3 gravity[2] are also non-degenerate.

However, for general e^a_μ, the extended vielbeins $(e^a_\mu,\, e^a_{\mu\nu})$ may be degenerate at some points. Even if this is the case, we can employ the usual method of the fibre bundles to avoid the singularity. Let us consider the case where the CS gauge field one-forms take the forms, $A = b^{-1}\, a\, b + b^{-1}\, db$, $\bar{A} = b\, \bar{a}\, b^{-1} + b\, db^{-1}$ with $b = \exp(r\, L_0)$[2]. r is the radial coordinate, and a and $\bar{a}$ are one-forms independent of r. Let the extended vielbeins be degenerate only at $r = r_0$. We cover the base manifold by two open coordinate neighborhoods, $U_1 = \{(r, t, \phi)|\, r > \alpha\}$, $U_2 = \{(r, t, \phi)|\, r < \beta\}$ with $r_0 < \alpha < \beta$. The gauge fields on U_i will be denoted as A^i and $\bar{A}^i$, and we set $A^1(r) \equiv A(r)$, $\bar{A}^1(r) \equiv \bar{A}(r)$. We choose as a transition function (gauge transformation) on the overlap $U_1 \cap U_2 = \{(r, t, \phi)|\, \alpha < r < \beta\}$, an extended local translation $V_- = \exp \Lambda_- = \exp(-y L_0)$, where L_0 is one of the $sl(3, R)$ generators (A.2) and y is a constant satisfying $y > \beta - r_0 > \beta - \alpha$. The transition function V_- has an effect of a translation $r \to r - y$ on the gauge fields $A^1(r)$ and $\bar{A}^1(r)$ in U_1, and the gauge fields on U_2 are given by $A^2(r) = A^1(r - y)$ and $\bar{A}^2(r) = \bar{A}^1(r - y)$. Then, we have $r - y < \beta - y < r_0$ in U_2 and the extended vielbeins $(e^a_\mu,\, e^a_{\mu\nu})$ computed from $A^2(r)$ and $\bar{A}^2(r)$ are non-degenerate in U_2. If there are more degenerate points, the extended vielbeins can also be made non-degenerate by the same procedure.

Now if e^a_μ and $e^a_{\mu\nu}$ are combined, these can be regarded as an 8×8 matrix and it has an inverse matrix. Let us define inverse vielbeins E^μ_a and $E^{\mu\nu}_a$ by

$$
\begin{aligned}
E^\mu_a\, e^a_\nu &= \delta^\mu_\nu, \qquad E^\mu_a\, e^a_{\nu\lambda} = 0,\\
E^{\mu\nu}_a\, e^a_\lambda &= 0, \qquad E^{\mu\nu}_a\, e^a_{\lambda\rho} = \delta^\mu_\lambda\, \delta^\nu_\rho + \delta^\nu_\lambda\, \delta^\mu_\rho - \frac{2}{3}\, g_{\lambda\rho}\, g^{\mu\nu}. && (3.7)
\end{aligned}
$$

The right-hand side of the last eq ensures the tracelessness of $E^{\mu\nu}_a$: $g_{\mu\nu}\, E^{\mu\nu}_a = 0$. They also

[4] It is easy to find background vielbeins for which actually the matrix $\rho \neq 0$.

satisfy the relation

$$e^a_\mu\, E^\mu_b + \frac{1}{2}\, e^a_{\mu\nu}\, E^{\mu\nu}_b = \delta^a{}_b. \tag{3.8}$$

3.2 New connections

Let us turn to the torsion-free condition (3.2) now expressed in terms of components.

$$\partial_\mu\, e_\nu - \partial_\nu\, e_\mu + [\omega_\mu, e_\nu] - [\omega_\nu,\, e_\mu] = 0 \tag{3.9}$$

This relation implies that $\partial_\mu\, e_\nu + [\omega_\mu, e_\nu]$ is symmetric for interchange of μ and ν. Because this is a traceless matrix, it can be expanded in terms of e_λ and $e_{\lambda\rho}$.

$$\partial_\mu\, e_\nu + [\omega_\mu, e_\nu] = \Gamma^\rho_{\mu\nu}\; e_\rho + \frac{1}{2}\; \Gamma^{\sigma\rho}_{\mu\nu}\; e_{\sigma\rho} \tag{3.10}$$

Here $\Gamma^\lambda_{\mu\nu}$ and $\Gamma^{\lambda\rho}_{\mu\nu}$ are two connections to be determined later, as functions of the metric-like fields and their derivatives. These are symmetric in the lower indices μ and ν. By definition, $\Gamma^{\lambda\rho}_{\mu\nu}$ must satisfy $g_{\lambda\rho}\, \Gamma^{\lambda\rho}_{\mu\nu} = 0$, since it is multiplied by $e_{\lambda\rho}$. The existence of $\Gamma^\lambda_{\mu\nu}$ and $\Gamma^{\lambda\rho}_{\mu\nu}$ are ensured by the assumed linear independence of $e_\mu = t_a\, e^a_\mu$ and $e_{\mu\nu} = t_a\, e^a_{\mu\nu}$. By multiplying both sides in the component form by E^λ_a and $E^{\lambda\rho}_a$, respectively, we can represent these connections in terms of ω^a_μ and the vielbeins. However, we would like to express them in terms of only the vielbeins.

In order to derive such expressions, let us multiply (3.10) by e_λ from the right.

$$\partial_\mu\, e_\nu\;\, e_\lambda + [\omega_\mu, e_\nu]\;\, e_\lambda = \Gamma^\rho_{\mu\nu}\;\, e_\rho\;\, e_\lambda + \frac{1}{2}\;\, \Gamma^{\sigma\rho}_{\mu\nu}\;\, e_{\sigma\rho}\;\, e_\lambda \tag{3.11}$$

Then by replacing ν by λ in (3.10), multiplying it by e_ν from the left and adding the result to (3.11) the following relation is obtained.

$$\begin{aligned} \partial_\mu\; (e_\nu\, e_\lambda) + [\omega_\mu,\, e_\nu\, e_\lambda] \;\; &= \;\; \Gamma^\sigma_{\mu\nu}\;\, e_\sigma\, e_\lambda + \Gamma^\sigma_{\mu\lambda}\;\, e_\nu\, e_\sigma \\ &\quad + \frac{1}{2}\, \Gamma^{\sigma\kappa}_{\mu\nu}\, e_{\sigma\kappa}\, e_\lambda + \frac{1}{2}\, \Gamma^{\sigma\kappa}_{\mu\lambda}\, e_\nu\, e_{\sigma\kappa} \end{aligned} \tag{3.12}$$

By taking the trace of this eq and using (2.10) the following relation is obtained.

$$\partial_\mu\, g_{\nu\lambda} = \Gamma^\sigma_{\mu\nu}\, g_{\sigma\lambda} + \Gamma^\sigma_{\mu\lambda}\, g_{\sigma\nu} + \frac{1}{2}\, \Gamma^{\sigma\kappa}_{\mu\nu}\;\, g_{(\sigma\kappa)\lambda} + \frac{1}{2}\, \Gamma^{\sigma\kappa}_{\mu\lambda}\;\, g_{(\sigma\kappa)\nu} \tag{3.13}$$

Here the following field is defined.

$$g_{(\mu\nu)\lambda} = \frac{1}{2}\, \mathrm{tr}\; (e_{\mu\nu}\, e_\lambda) \tag{3.14}$$

The parenthesis in the subscript of g on the left-hand side is put to make the permutation symmetry manifest. This field is different from, but related to, the spin-3 gauge field $\phi_{\mu\nu\lambda}$ defined in (2.11).

$$\phi_{\mu\nu\lambda} = g_{(\mu\nu)\lambda} + \frac{1}{6}\, g_{\mu\nu}\, \mathrm{tr}(\rho\, e_\lambda) \tag{3.15}$$

This new field satisfies $g^{\mu\nu}\, g_{(\mu\nu)\lambda} = 0$.[5] The matrix ρ was defined in (3.6). Note that the trace in the second term on the right can be represented in terms of the spin-three gauge field.

$$\mathrm{tr}(\rho\, e_\lambda) = 2\, \phi_{\mu\nu\lambda}\, g^{\mu\nu} = 2\, \phi^{\mu}{}_{\mu\lambda} \tag{3.16}$$

As shown in the above expression spacetime indices are raised and/or lowered by $g^{\mu\nu}$ and $g_{\mu\nu}$.[6]

Now eq (3.13) can be solved to yield the following relation by using the usual method.

$$\Gamma^{\lambda}_{\mu\nu} + \frac{1}{2}\, \Gamma^{\sigma\kappa}_{\mu\nu}\, \phi_{\sigma\kappa}{}^{\lambda} = \frac{1}{2}\, g^{\lambda\rho}\, (\partial_\mu\, g_{\nu\rho} + \partial_\nu\, g_{\mu\rho} - \partial_\rho\, g_{\mu\nu}) \equiv \hat{\Gamma}^{\lambda}_{\mu\nu} \tag{3.17}$$

$\hat{\Gamma}^{\lambda}_{\mu\nu}$ is the ordinary Christoffel symbol (connection).

3.3 Determination of $\Gamma^{\lambda\rho}_{\mu\nu}$

Another relation is required to determine the two connections. Multiplying (3.12) by e_ρ from the right and adding a similar eq obtained by multiplying (3.10) (with replacement $\nu \to \rho$) by $e_\nu e_\lambda$ from the left the following eq is obtained.

$$\begin{aligned} & \partial_\mu\, (e_\nu\, e_\lambda\, e_\rho) + [\omega_\mu, e_\nu\, e_\lambda\, e_\rho] \\ = \; & \Gamma^{\sigma}_{\mu\nu}\; e_\sigma\, e_\lambda\, e_\rho + \Gamma^{\sigma}_{\mu\lambda}\; e_\nu\, e_\sigma\, e_\rho + \Gamma^{\sigma}_{\mu\rho}\; e_\nu\, e_\lambda\, e_\sigma \\ & + \frac{1}{2}\; \Gamma^{\sigma\kappa}_{\mu\nu}\, e_{\sigma\kappa}\, e_\lambda\, e_\rho + \frac{1}{2}\; \Gamma^{\sigma\kappa}_{\mu\lambda}\, e_\nu\, e_{\sigma\kappa}\, e_\rho + \frac{1}{2}\; \Gamma^{\sigma\kappa}_{\mu\rho}\, e_\nu\, e_\lambda\, e_{\sigma\kappa} \end{aligned} \tag{3.18}$$

Now interchange λ and ρ in the above eq and add the result to the above. Taking the trace leads to the following eq.

$$\begin{aligned} & \partial_\mu\; \mathrm{tr}\; e_\nu\, \{e_\lambda,\, e_\rho\} \\ = \; & \Gamma^{\sigma}_{\mu\nu}\; \mathrm{tr}\; e_\sigma\, \{e_\lambda,\, e_\rho\} + \Gamma^{\sigma}_{\mu\lambda}\; \mathrm{tr}\; e_\nu\, \{e_\sigma,\, e_\rho\} + \Gamma^{\sigma}_{\mu\rho}\; \mathrm{tr}\; e_\nu\, \{e_\lambda,\, e_\sigma\} \\ & + \frac{1}{2}\; \Gamma^{\sigma\kappa}_{\mu\nu}\, \mathrm{tr}\; e_{\sigma\kappa}\, \{e_\lambda,\, e_\rho\} + \frac{1}{2}\; \Gamma^{\sigma\kappa}_{\mu\lambda}\, \mathrm{tr}\; e_{\sigma\kappa}\, \{e_\rho,\, e_\nu\} + \frac{1}{2}\; \Gamma^{\sigma\kappa}_{\mu\rho}\, \mathrm{tr}\; e_{\sigma\kappa}\, \{e_\lambda,\, e_\nu\} \end{aligned} \tag{3.19}$$

In the above eq let us note that one can make the following substitutions.

$$\mathrm{tr}\, e_\sigma\; \{e_\lambda,\, e_\rho\} = 4\, \phi_{\sigma\lambda\rho}, \tag{3.20}$$

$$\mathrm{tr}\, e_{\sigma\kappa}\, \{e_\lambda,\, e_\rho\} = 4\, g_{(\sigma\kappa)(\lambda\rho)} + \frac{2}{3}\, g_{\lambda\rho}\; \mathrm{tr}\; (e_{\sigma\kappa}\, \rho) \tag{3.21}$$

[5]So in the text, when $g^{\mu\nu}$ exists, $\phi_{\mu\nu\lambda}$ and $g_{(\mu\nu)\lambda}$ will be interchanged without notice.
[6]Exception: the indices of e^a_μ, $e^a_{\mu\nu}$, E^μ_a and $E^{\mu\nu}_a$ will not be raised and/or lowered by the metric tensors.

Here $g_{(\sigma\kappa)(\lambda\rho)}$ is defined in appendix C. Furthermore, the connection $\Gamma^{\lambda}_{\mu\nu}$ can be eliminated by use of (3.17). This yields the eq.

$$\begin{aligned}\hat{\nabla}_\mu\ \phi_{\nu\lambda\rho} &= \frac{1}{2}\Gamma^{\sigma\kappa}_{\mu\nu}\left(g_{(\sigma\kappa)(\lambda\rho)} - g_{(\sigma\kappa)}{}^{\tau}\, g_{\tau(\lambda\rho)} - \frac{1}{6}\, g_{\lambda\rho}\, g_{(\sigma\kappa)}{}^{\tau}\,\mathrm{tr}\,(\rho\, e_\tau) + \frac{1}{6}\, g_{\lambda\rho}\ \mathrm{tr}\,(\rho e_{\sigma\kappa})\right) \\ &\quad +(\text{cyclic permutations of } \nu,\lambda,\rho) \end{aligned} \tag{3.22}$$

Here $\hat{\nabla}_\mu$ is the covariant derivative which uses the Christoffel connection.

Let us introduce a 5×5 matrix $M_{(\sigma\kappa)(\lambda\rho)}$ by

$$M_{(\sigma\kappa)(\lambda\rho)} \equiv g_{(\sigma\kappa)(\lambda\rho)} - g_{(\sigma\kappa)}{}^{\tau}\, g_{\tau(\lambda\rho)}. \tag{3.23}$$

Its inverse matrix $J^{(\lambda\rho)(\sigma\kappa)}$ is assumed to exist[7] and defined by the following eq.

$$\frac{1}{2}\ M_{(\mu\nu)(\lambda\rho)}\ J^{(\lambda\rho)(\sigma\kappa)} = \delta^\sigma_\mu\delta^\kappa_\nu + \delta^\sigma_\nu\delta^\kappa_\mu - \frac{2}{3}\ g_{\mu\nu}\ g^{\sigma\kappa} \tag{3.24}$$

Then (3.22) is rewritten as

$$\begin{aligned}\hat{\nabla}_\mu\ \phi_{\nu\lambda\rho} &= \frac{1}{2}\Gamma^{\sigma\kappa}_{\mu\nu}\ M_{(\sigma\kappa)(\lambda\rho)} + \frac{1}{2}\Gamma^{\sigma\kappa}_{\mu\lambda}\ M_{(\sigma\kappa)(\nu\rho)} + \frac{1}{2}\Gamma^{\sigma\kappa}_{\mu\rho}\ M_{(\sigma\kappa)(\nu\lambda)} \\ &\quad +\frac{1}{12}\ \left(g_{\lambda\rho}\ \Gamma^{\sigma\kappa}_{\mu\nu} + g_{\nu\rho}\ \Gamma^{\sigma\kappa}_{\mu\lambda} + g_{\lambda\nu}\ \Gamma^{\sigma\kappa}_{\mu\rho}\right)\ W_{\sigma\kappa} \end{aligned} \tag{3.25}$$

Here $W_{\sigma\kappa}$ is defined by

$$W_{\sigma\kappa} = \mathrm{tr}\,(e_{\sigma\kappa} - g_{(\sigma\kappa)}{}^{\tau}\ e_\tau)\,\rho. \tag{3.26}$$

We now introduce the following field.

$$\Phi_{\nu\lambda\rho} = \phi_{\nu\lambda\rho} - \frac{1}{5}\,(g_{\lambda\rho}\,\phi_{\nu\sigma}{}^{\sigma} + g_{\rho\nu}\,\phi_{\lambda\sigma}{}^{\sigma} + g_{\nu\lambda}\,\phi_{\rho\sigma}{}^{\sigma}) \tag{3.27}$$

This tensor is traceless for each pair of indices, $g^{\lambda\rho}\,\Phi_{\nu\lambda\rho} = 0$. In terms of this field, we can derive the following eq from (3.25).

$$\begin{aligned}\hat{\nabla}_\mu\,\Phi_{\nu\lambda\rho} &= \frac{1}{2}\Gamma^{\sigma\kappa}_{\mu\nu}\ M_{(\sigma\kappa)(\lambda\rho)} - \frac{1}{5}\, g_{\lambda\rho}\, g^{\tau\eta}\,\Gamma^{\sigma\kappa}_{\mu\tau}\, M_{(\sigma\kappa)(\eta\nu)} \\ &\quad + \text{cyclic permutations of }\ (\nu,\lambda,\rho) \end{aligned} \tag{3.28}$$

This eq determines $\Gamma\, M$.

$$\begin{aligned}\Gamma^{\sigma\kappa}_{\mu\nu}\, M_{(\sigma\kappa)(\lambda\rho)} &= \frac{2}{3}\ \left(\hat{\nabla}_\mu\ \Phi_{\nu\lambda\rho} + \hat{\nabla}_\nu\ \Phi_{\mu\lambda\rho}\right) - \frac{1}{3}\ \left(\hat{\nabla}_\lambda\ \Phi_{\mu\nu\rho} + \hat{\nabla}_\rho\ \Phi_{\mu\nu\lambda}\right) \\ &\quad +\frac{2}{9}\, g_{\lambda\rho}\,\hat{\nabla}^{\kappa}\,\Phi_{\mu\nu\kappa} + S_{\mu\nu,\lambda\rho} \end{aligned} \tag{3.29}$$

Here $S_{\mu\nu,(\lambda\rho)}$ is a function which satisfies

$$\begin{aligned} &S_{\mu\nu,\lambda\rho} = S_{\nu\mu,\lambda\rho} = S_{\mu\nu,\rho\lambda}, \qquad g^{\lambda\rho}\, S_{\mu\nu,\lambda\rho} = 0, \\ &S_{\mu\nu,\lambda\rho} - \frac{2}{5}\, g_{\lambda\rho}\, g^{\tau\eta}\, S_{\mu\tau,\eta\nu} + \text{cyclic permutations of } (\nu,\lambda,\rho) = 0. \end{aligned} \tag{3.30}$$

[7]This is true for AdS_3 solution as shown in (B.10).

By using the matrix J (3.24), (3.29) gives $\Gamma^{\lambda\rho}_{\mu\nu}$.

$$\Gamma^{\lambda\rho}_{\mu\nu} \quad = \quad \frac{1}{6}\,(\hat{\nabla}_\mu\;\Phi_{\nu\kappa\sigma} + \hat{\nabla}_\nu\;\Phi_{\mu\kappa\sigma} - \hat{\nabla}_\kappa\;\Phi_{\mu\nu\sigma})\;J^{(\kappa\sigma)(\lambda\rho)} + \frac{1}{4}\,S_{\mu\nu,\kappa\sigma}\,J^{(\kappa\sigma)(\lambda\rho)} \tag{3.31}$$

Finally, by contracting (3.25) with $g^{\lambda\rho}$ one obtains another algebraic eq which $S_{\mu\nu,(\lambda\rho)}$ must satisfy.

$$\begin{aligned}
\hat{\nabla}_\mu\,\phi_{\nu\lambda}{}^{\lambda} &= \Gamma^{\sigma\kappa}_{\mu\lambda}\,M_{(\sigma\kappa)(\nu\rho)}\,g^{\lambda\rho} + \frac{5}{12}\,\Gamma^{\sigma\kappa}_{\mu\nu}\,W_{\sigma\kappa} \\
&= g^{\lambda\rho}\,S_{\mu\lambda,\nu\rho} + \frac{5}{48}\,S_{\mu\nu,\alpha\beta}\,J^{(\alpha\beta)(\sigma\kappa)}\,W_{\sigma\kappa} + \frac{5}{9}\,\hat{\nabla}^\lambda\,\Phi_{\mu\nu\lambda} \\
&\quad + \frac{5}{72}\,(\hat{\nabla}_\mu\,\Phi_{\nu\alpha\beta} + \hat{\nabla}_\nu\,\Phi_{\mu\alpha\beta} - \hat{\nabla}_\alpha\,\Phi_{\mu\nu\beta})\,J^{(\alpha\beta)(\sigma\kappa)}\,W_{\sigma\kappa}
\end{aligned} \tag{3.32}$$

The eqs (3.31) and (3.17) determine $\Gamma^{\lambda}_{\mu\nu}$.

$$\Gamma^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu} - \frac{1}{12}\,\left(\hat{\nabla}_\mu\;\Phi_{\nu\kappa\sigma} + \hat{\nabla}_\nu\;\Phi_{\mu\kappa\sigma} - \hat{\nabla}_\kappa\;\Phi_{\mu\nu\sigma} + \frac{3}{2}\,S_{\mu\nu,\kappa\sigma}\right)\;J^{(\kappa\sigma)(\rho\tau)}\;\phi_{\rho\tau}{}^{\lambda} \tag{3.33}$$

The explicit solution for $S_{\mu\nu,\lambda\rho}$ is much involved and is presented in appendix D.

4 Spin connection as a solution to the torsion-free condition

In this section the torsion-free condition (3.10) will be solved for the spin connection ω_μ. For this purpose covariant derivative of the vielbein will be introduced. This must be done in such a way that the derivative is compatible with the metric $g_{\mu\nu}$ and other gauge fields $g_{(\mu\nu)\lambda}$, $g_{(\mu\nu)(\lambda\rho)}$.

4.1 Covariant derivatives

The torsion-free condition (3.10) takes the form of covariant constancy of e^a_ν.

$$D_\mu\;e^a_\nu \quad \equiv \quad \nabla_\mu\;e^a_\nu + f^a{}_{bc}\;\omega^b_\mu\;e^c_\nu = 0, \tag{4.1}$$

$$\nabla_\mu\;e^a_\nu \quad \equiv \quad \partial_\mu\;e^a_\nu - \Gamma^\lambda_{\mu\nu}\;e^a_\lambda - \frac{1}{2}\;\Gamma^{\lambda\rho}_{\mu\nu}\;e^a_{\lambda\rho} \tag{4.2}$$

The first eq is defining a full covariant derivative D_μ and the second eq is defining ∇_μ. Note that ∇_μ is a new covariant derivative which differs from $\hat{\nabla}_\mu$ associated with the Christoffel symbol $\hat{\Gamma}^\lambda_{\mu\nu}$. The last term of (4.2) can be rewritten as $-(1/4)\,\Gamma^{\lambda\rho}_{\mu\nu}\,d^a{}_{bc}\,e^b_\lambda\,e^c_\rho$. This definition keeps the covariance under the local frame rotations.

By using (3.17) it can be shown that the effect of ∇_μ on $g_{\nu\lambda}$ agrees with that of $\hat{\nabla}_\mu$.

$$\begin{aligned}
\nabla_\mu\,g_{\nu\lambda} &= \nabla_\mu\,(e^a_\nu\,e_{a\lambda}) = (\nabla_\mu\,e^a_\nu)\,e_{a\lambda} + e^a_\nu\,(\nabla_\mu\,e_{a\lambda}) \\
&= \partial_\mu\,g_{\nu\lambda} - \Gamma^\rho_{\mu\nu}\,g_{\rho\lambda} - \Gamma^\rho_{\mu\lambda}\,g_{\rho\nu} - \frac{1}{2}\,\Gamma^{\rho\sigma}_{\mu\nu}\,g_{(\rho\sigma)\lambda} - \frac{1}{2}\,\Gamma^{\rho\sigma}_{\mu\lambda}\,g_{(\rho\sigma)\nu} = 0.
\end{aligned} \tag{4.3}$$

Note that this can be shown by using only (3.17). The explicit expression for $\Gamma^{\rho\sigma}_{\mu\nu}$ is unnecessary.

Let us next start with (3.12). By interchanging ν and λ and adding the result to (3.12) one obtains an eq

$$\begin{aligned}\partial_\mu \, \{e_\nu,\, e_\lambda\} + [\omega_\mu,\, \{e_\nu,\, e_\lambda\}] &= \Gamma^\sigma_{\mu\nu} \, \{e_\sigma,\, e_\lambda\} + \Gamma^\sigma_{\mu\lambda} \, \{e_\nu,\, e_\sigma\} + \frac{1}{2}\, \Gamma^{\sigma\kappa}_{\mu\nu} \, \{e_{\sigma\kappa},\, e_\lambda\} \\ &\quad + \frac{1}{2}\, \Gamma^{\sigma\kappa}_{\mu\lambda} \, \{e_\nu,\, e_{\sigma\kappa}\}. \end{aligned} \tag{4.4}$$

Let us subtract the terms proportional to the identity matrix from the above eq. The trace part was already studied just after (3.12). Owing to (3.5) and (3.14) the result is

$$\begin{aligned}&\partial_\mu \, e_{\nu\lambda} + [\omega_\mu,\, e_{\nu\lambda}] \\ &= -\frac{1}{3}\, \partial_\mu \, (g_{\nu\lambda} \, \rho) - \frac{1}{3}\, g_{\nu\lambda} \, [\omega_\mu,\, \rho] + \Gamma^\sigma_{\mu\nu} \, e_{\sigma\lambda} + \frac{1}{3}\, \Gamma^\sigma_{\mu\nu} \, g_{\sigma\lambda} \, \rho + \Gamma^\sigma_{\mu\lambda} \, e_{\sigma\nu} + \frac{1}{3}\, \Gamma^\sigma_{\mu\lambda} \, g_{\sigma\nu} \, \rho \\ &+ \frac{1}{4}\, \Gamma^{\sigma\kappa}_{\mu\nu} \, \{e_{\sigma\kappa},\, e_\lambda\} - \frac{1}{3}\, \Gamma^{\sigma\kappa}_{\mu\nu} \, g_{(\sigma\kappa)\lambda} \, \boldsymbol{I} + \frac{1}{4}\, \Gamma^{\sigma\kappa}_{\mu\lambda} \, \{e_{\sigma\kappa},\, e_\nu\} - \frac{1}{3}\, \Gamma^{\sigma\kappa}_{\mu\lambda} \, g_{(\sigma\kappa)\nu} \, \boldsymbol{I} \end{aligned} \tag{4.5}$$

Expansion in terms of the basis t_a yields the eq.

$$\begin{aligned}&f^a{}_{bc} \, \omega^b{}_\mu \, e^c{}_{\nu\lambda} \\ &= -\partial_\mu \, e^a{}_{\nu\lambda} + \Gamma^\sigma_{\mu\nu} \, e^a_{\sigma\lambda} + \Gamma^\sigma_{\mu\lambda} \, e^a_{\sigma\nu} + \frac{1}{4}\, \Gamma^{\sigma\kappa}_{\mu\nu} \, d^a{}_{bc} \, e^b_{\sigma\kappa} \, e^c_\lambda + \frac{1}{4}\, \Gamma^{\sigma\kappa}_{\mu\lambda} \, d^a{}_{bc} \, e^b_{\sigma\kappa} \, e^c_\nu \\ &- \frac{1}{3}\, \rho^a \, \partial_\mu \, g_{\nu\lambda} - \frac{1}{3}\, g_{\nu\lambda} \, \partial_\mu \rho^a - \frac{1}{3}\, g_{\nu\lambda} \, f^a{}_{bc} \, \omega^b_\mu \, \rho^c + \frac{1}{3}\, g_{\lambda\sigma} \, \Gamma^\sigma_{\mu\nu} \, \rho^a + \frac{1}{3}\, g_{\nu\sigma} \, \Gamma^\sigma_{\mu\lambda} \, \rho^a \end{aligned} \tag{4.6}$$

Here ρ^a is defined by $\rho = \rho^a \, t_a$. Contraction of the left-hand side with $g^{\nu\lambda}$ vanishes. So the same must hold for the right-hand side. This leads to a differential eq for ρ. This eq can be interpreted as the covariant-constancy condition for ρ^a.

$$D_\mu \, \rho^a \equiv \nabla_\mu \, \rho^a + f^a{}_{bc} \, \omega^b_\mu \, \rho^c = 0, \tag{4.7}$$

$$\nabla_\mu \, \rho^a \equiv \partial_\mu \, \rho^a + g^{\nu\lambda} \, \Gamma^{\kappa\sigma}_{\mu\nu} \, \left(g_{(\kappa\sigma)}{}^\rho \, e^a_{\rho\lambda} - \frac{1}{2}\, d^a{}_{bc} \, e^b_{\sigma\kappa} \, e^c_\lambda \right) + \frac{1}{3}\, \rho^a \, \Gamma^{\sigma\kappa}_{\mu\nu} \, g_{(\sigma\kappa)}{}^\nu \tag{4.8}$$

The first eq is defining a full covariant derivative $D_\mu \, \rho^a$ in terms of $\nabla_\mu \, \rho^a$ and the second eq is defining $\nabla_\mu \, \rho^a$. Here (3.17) is used to relate $\Gamma^\nu_{\mu\nu}$ to $\Gamma^{\sigma\kappa}_{\mu\nu}$.

The remaining traceless part of (4.6) can also be interpreted as expressing the property of covariant-constancy of $e^a_{\nu\lambda}$.

$$D_\mu \, e^a_{\nu\lambda} \equiv \nabla_\mu \, e^a_{\nu\lambda} + f^a{}_{bc} \, \omega^b_\mu \, e^c_{\nu\lambda} = 0, \tag{4.9}$$

$$\begin{aligned}\nabla_\mu \, e^a_{\nu\lambda} \equiv\ & \partial_\mu \, e^a_{\nu\lambda} - \Gamma^\sigma_{\mu\nu} \, e^a_{\sigma\lambda} - \Gamma^\sigma_{\mu\lambda} \, e^a_{\nu\sigma} - \frac{1}{4}\, \Gamma^{\sigma\kappa}_{\mu\nu} \, d^a{}_{bc} \, e^b_{\sigma\kappa} \, e^c_\lambda - \frac{1}{4}\, \Gamma^{\sigma\kappa}_{\mu\lambda} \, d^a{}_{bc} \, e^b_{\sigma\kappa} \, e^c_\nu \\ & + g_{\nu\lambda} \, \left(\frac{1}{6}\, \Gamma^{\sigma\kappa}_{\mu\rho} \, d^a{}_{bc} \, e^b_{\sigma\kappa} \, e^c_\tau \, g^{\rho\tau} - \frac{1}{3}\, \Gamma^{\tau\eta}_{\mu\rho} \, g_{(\tau\eta)}{}^\sigma \, g^{\kappa\rho} \, e^a_{\sigma\kappa} - \frac{1}{9}\, \rho^a \, \Gamma^{\sigma\kappa}_{\mu\tau} \, g_{(\sigma\kappa)}{}^\tau \right) \\ & + \frac{1}{6}\, \rho^a \, \left(\Gamma^{\sigma\kappa}_{\mu\nu} \, g_{(\sigma\kappa)\lambda} + \Gamma^{\sigma\kappa}_{\mu\lambda} \, g_{(\sigma\kappa)\nu} \right) \end{aligned} \tag{4.10}$$

The righthand side of (4.10) can be expanded in terms of e^a_μ and $e^a_{\mu\nu}$ as

$$\nabla_\mu\, e^a_{\nu\lambda} = \partial_\mu\; e^a_{\nu\lambda} - \Gamma^\rho_{\mu,(\nu\lambda)}\, e^a_\rho - \frac{1}{2}\,\Gamma^{\rho\sigma}_{\mu,(\nu\lambda)}\, e^a_{\rho\sigma} \tag{4.11}$$

and a new set of connections are defined.

$$\begin{aligned}
\Gamma^\rho_{\mu,(\nu\lambda)} \quad &\equiv \quad E^\rho_a\,(\partial_\mu\, e^a_{\nu\lambda} - \nabla_\mu\, e^a_{\nu\lambda}) = \frac{1}{4}\,\Gamma^{\sigma\kappa}_{\mu\nu}\, d^a{}_{bc}\, E^\rho_a\, e^b_{\sigma\kappa}\, e^c_\lambda - \frac{1}{12}\, g_{\nu\lambda}\,\Gamma^{\sigma\kappa}_{\mu\eta}\, d^a{}_{bc}\, E^\rho_a\, e^b_{\sigma\kappa}\, e^c_\tau\, g^{\eta\tau} \\
&+\frac{1}{18}\, g_{\nu\lambda}\,\rho^a\, E^\rho_a\,\Gamma^{\sigma\kappa}_{\mu\tau}\, g_{(\sigma\kappa)}{}^\tau - \frac{1}{6}\,\rho^a\, E^\rho_a\,\Gamma^{\sigma\kappa}_{\mu\nu}\, g_{(\sigma\kappa)\lambda} + (\nu \leftrightarrow \lambda), \qquad (4.12)\\
\Gamma^{(\rho\sigma)}_{\mu,(\nu\lambda)} \quad &\equiv \quad E^{\rho\sigma}_a\,(\partial_\mu\, e^a_{\nu\lambda} - \nabla_\mu\, e^a_{\nu\lambda}) = \Gamma^\rho_{\mu\nu}\,\delta^\sigma_\lambda + \Gamma^\sigma_{\mu\nu}\,\delta^\rho_\lambda - \frac{2}{3}\,\Gamma^\kappa_{\mu\nu}\, g_{\kappa\lambda}\, g^{\rho\sigma} + \frac{1}{4}\,\Gamma^{\kappa\eta}_{\mu\nu}\, d^a{}_{bc}\, E^{\rho\sigma}_a\, e^b_{\kappa\eta}\, e^c_\lambda \\
&-\frac{1}{12}\, g_{\nu\lambda}\,\Gamma^{\beta\kappa}_{\mu\alpha}\, d^a{}_{bc}\, E^{\rho\sigma}_a\, e^b_{\beta\kappa}\, e^c_\tau\, g^{\alpha\tau} + \frac{1}{6}\, g_{\nu\lambda}\,\Gamma^{\tau\eta}_{\mu\alpha}\, g_{(\tau\eta)}{}^\rho g^{\sigma\alpha} \\
&+\frac{1}{6}\, g_{\nu\lambda}\,\Gamma^{\tau\eta}_{\mu\alpha}\, g_{(\tau\eta)}{}^\sigma\, g^{\rho\alpha} - \frac{1}{9}\, g_{\nu\lambda}\,\Gamma^{\tau\eta}_{\mu\alpha}\, g_{(\eta\tau)}{}^\alpha\, g^{\rho\sigma} + \frac{1}{18}\, g_{\nu\lambda}\,\rho^a\, E^{\rho\sigma}_a\,\Gamma^{\alpha\beta}_{\mu\tau}\, g_{(\alpha\beta)}{}^\tau \\
&-\frac{1}{6}\,\rho^a\, E^{\rho\sigma}_a\,\Gamma^{\alpha\beta}_{\mu\nu}\, g_{(\alpha\beta)\lambda} + (\nu \leftrightarrow \lambda) \qquad (4.13)
\end{aligned}$$

In the above eqs, $\rho\, E$'s are given by

$$\begin{aligned}
\rho^a\, E^\mu_a \quad &= \quad \phi^{\mu\lambda}{}_\lambda + \frac{1}{4}\, g^\mu{}_{(\tau\eta)}\, J^{(\tau\eta)(\sigma\kappa)}\,\Big(\phi_{\sigma\kappa}{}^\nu\,\phi_{\nu\lambda}{}^\lambda - \phi_{(\sigma\kappa)(\lambda\rho)}\, g^{\lambda\rho}\Big), \qquad (4.14)\\
\rho^a\, E^{\mu\nu}_a \quad &= \quad \frac{1}{2}\, J^{(\mu\nu)(\lambda\rho)}\,(\phi_{(\lambda\rho)(\kappa\sigma)} - \phi_{\lambda\rho\tau}\,\phi^\tau{}_{\kappa\sigma})\, g^{\kappa\sigma} \qquad (4.15)
\end{aligned}$$

To prove these eqs, $\rho^a = \frac{1}{2}\, g^{\mu\nu}\, d^a{}_{bc}\, e^b_\mu\, e^c_\nu$ and (C.2), (E.4)-(E.6) and (E.11) must be used. Furthermore, $d^a{}_{bc}$ can be replaced by (C.6) in appendix C. In this way the righthand sides of (4.12), (4.13) could be expressed solely in terms of the metric-like fields. This will, however, not be attempted in this paper to avoid complication.

Alternatively, the covariant derivative of $e^a_{\nu\lambda}$ (4.10) can be derived from that for e^a_ν (4.2) by using $e^a_{\nu\lambda} = \frac{1}{2}\, d^a{}_{bc}\, e^b_\nu\, e^c_\lambda - \frac{1}{3}\, g_{\nu\lambda}\,\rho^a$. Therefore, we can also write

$$\nabla_\mu\, e^a_{\nu\lambda} = \frac{1}{2}\, d^a{}_{bc}\,(e^b_\nu\,\nabla_\mu\, e^c_\lambda + e^b_\lambda\,\nabla_\mu\, e^c_\nu) - \frac{1}{3}\, g_{\nu\lambda}\,\nabla_\mu\,\rho^a, \tag{4.16}$$

For ρ^a, we can write

$$\nabla_\mu\,\rho^a = d^a{}_{bc}\, g^{\nu\lambda}\, e^b_\nu\,\nabla_\mu\, e^c_\lambda. \tag{4.17}$$

Because $D_\mu\, d^a{}_{bc} = 0$[8] and $D_\mu\, g^{\lambda\rho} = -g^{\lambda\kappa}\,(\nabla_\mu\, g_{\kappa\sigma})\, g^{\sigma\rho} = 0$, the covariant constancy of ρ^a is a result of that of e^a_μ.

4.2 Covariant derivatives for general tensors

Let v^a be an arbitrary vector in the local Lorentz frame. This vector can be expanded in terms of the vielbeins in either way

$$v^a = v^\mu\, e^a_\mu + \frac{1}{2}\, v^{(\mu\nu)}\, e^a_{\mu\nu}, \tag{4.18}$$

[8]This can be proved by using Jacobi's identity containing d_{abc} and f_{abc}.

or

$$v^a = v_\mu\, E^{a\mu} + \frac{1}{2}\, v_{(\mu\nu)}\, E^{a\mu\nu}. \tag{4.19}$$

(4.18) defines the contravariant components and (4.19) the covariant ones. $v^{(\mu\nu)}$ and $v_{(\mu\nu)}$ obey constraints $v^{(\mu\nu)}\, g_{\mu\nu} = 0$ and $v_{(\mu\nu)}\, g^{\mu\nu} = 0$, respectively. Similar decomposition can be performed for arbitrary tensors with an arbitrary number of local frame indices a's. A general rule for decomposition is that for each local frame index a there corresponds a pair of spacetime indices, μ and $(\mu\nu)$. The parenthesis in $(\mu\nu)$ implies the symmetry under interchange of μ and ν. To avoid confusion, the indices of v^μ, $v^{(\mu\nu)}$ will not be raised or lowered by $g_{\mu\nu}$.[9]

The covariant derivative of v^a is by definition given by $\nabla_\mu\, v^a = \partial_\mu\, v^a$. By use of (4.18) this leads to the identity.

$$\begin{aligned} & (\nabla_\mu\, v^\lambda)\, e^a_\lambda + v^\lambda\, (\nabla_\mu\, e^a_\lambda) + \frac{1}{2}\, (\nabla_\mu\, v^{(\lambda\rho)})\, e^a_{\lambda\rho} + \frac{1}{2}\, v^{(\lambda\rho)}\, (\nabla_\mu\, e^a_{\lambda\rho}) \\ = \; & \partial_\mu\, v^\lambda\, e^a_\lambda + v^\lambda\, (\partial_\mu\, e^a_\lambda) + \frac{1}{2}\, (\partial_\mu\, v^{(\lambda\rho)})\, e^a_{\lambda\rho} + \frac{1}{2}\, v^{(\lambda\rho)}\, (\partial_\mu\, e^a_{\lambda\rho}) \end{aligned} \tag{4.20}$$

By comparing the coefficients of e^a_ν and $e^a_{\nu\lambda}$ on both sides one obtains the definitions of the covariant derivatives of v^μ and $v^{(\mu\nu)}$.

$$\nabla_\mu\, v^\nu = \partial_\mu\, v^\nu + \Gamma^\nu_{\mu\lambda}\, v^\lambda + \frac{1}{2}\, \Gamma^\nu_{\mu,(\lambda\rho)}\, v^{(\lambda\rho)}, \tag{4.21}$$

$$\nabla_\mu\, v^{(\nu\lambda)} \;=\; \partial_\mu\, v^{(\nu\lambda)} + \Gamma^{\nu\lambda}_{\mu\rho}\, v^\rho + \frac{1}{2}\, \Gamma^{\nu\lambda}_{\mu,(\rho\sigma)}\, v^{(\rho\sigma)} \tag{4.22}$$

By using the expansion (4.19) and taking the similar steps, the covariant derivatives of the covariant components are also obtained.

$$\nabla_\mu\, v_\nu = \partial_\mu\, v_\nu - \Gamma^\lambda_{\mu\nu}\, v_\lambda - \frac{1}{2}\, \Gamma^{\lambda\rho}_{\mu\nu}\, v_{(\lambda\rho)}, \tag{4.23}$$

$$\nabla_\mu\, v_{(\nu\lambda)} \;=\; \partial_\mu\, v_{(\nu\lambda)} - \Gamma^\rho_{\mu,(\nu\lambda)}\, v_\rho - \frac{1}{2}\, \Gamma^{\rho\sigma}_{\mu,(\nu\lambda)}\, v_{(\rho\sigma)} \tag{4.24}$$

An extension to the covariant derivatives for the tensors with more indices will be straightforward and clear. The rule is the same as in the spin-2 gravity. For instance, by using (4.23) $\nabla_\mu\, g_{\nu\lambda}$ is calculated as

$$\nabla_\mu\, g_{\nu\lambda} = \partial_\mu\, g_{\nu\lambda} - \Gamma^\rho_{\mu\nu}\, g_{\rho\lambda} - \frac{1}{2}\, \Gamma^{\rho\sigma}_{\mu\nu}\, g_{(\rho\sigma)\lambda} - \Gamma^\rho_{\mu\lambda}\, g_{\nu\rho} - \frac{1}{2}\, \Gamma^{\rho\sigma}_{\mu\lambda}\, g_{\nu(\rho\sigma)}, \tag{4.25}$$

because $g_{(\rho\sigma)\lambda}$ is paired with $g_{\rho\lambda}$. This vanishes as in (4.3). This is an important property of the metric tensor G_{MN} introduced in appendix E, which can be used to raise and lower

[9] Instead, this will be done in terms of G_{MN} defined in appendix E.

the indices, μ and $(\mu\nu)$. One can also explicitly check eq $\nabla_\mu\, g_{\nu(\lambda\rho)} = 0$ by using (3.22), (4.12) and (4.13).

As we have seen, the indices of tensors have the structure, $\mu, (\mu\nu)$: general tensors can be written as $T_{MN\ldots}{}^{L\ldots}$, where M, N, L take two types of indices, μ, $(\mu\nu)$.

The covariant derivatives of the inverse vielbeins E are special examples of the above ones. They can be obtained by using the definitions (4.21) -(4.22).

$$\nabla_\mu\, E_a^\nu \;=\; \partial_\mu\, E_a^\nu + \Gamma^\nu_{\mu\lambda}\, E_a^\lambda + \frac{1}{2}\,\Gamma^\nu_{\mu,(\lambda\rho)}\, E_a^{\lambda\rho}, \tag{4.26}$$

$$\nabla_\mu\, E_a^{\nu\lambda} \;=\; \partial_\mu\, E_a^{\nu\lambda} + \Gamma^{\nu\lambda}_{\mu\rho}\, E_a^\rho + \frac{1}{2}\,\Gamma^{\nu\lambda}_{\mu,(\rho\sigma)}\, E_a^{\rho\sigma} \tag{4.27}$$

Although there exist $g_{\mu\nu}$ and $g^{\lambda\rho}$ on the righthand sides of (3.7), the above eqs are compatible with (3.7), because one has $\nabla_\mu\,(e_N^a\, E_a^M) = (\nabla_\mu\, e_N^a)\, E_a^M + e_N^a\,(\nabla_\mu\, E_a^M) = \partial_\mu\,(e_N^a\, E_a^M)$ and $\nabla_\mu\, \delta_M^N = \partial_\mu\, \delta_M^N - \Gamma^K_{\mu M}\, \delta_K^N + \Gamma^N_{\mu K}\, \delta_M^K = \partial_\mu\, \delta_M^N$, where δ_M^N is defined around (E.15) in appendix E.

4.3 Spin connection

It is now easy to solve for ω_μ^a. Multiplication of (4.1) and (4.9) by E_d^ν and $\frac{1}{2}\, E_d^{\nu\lambda}$, respectively, and adding the two we obtain the spin connection in the adjoint representation.

$$\omega^a_{\mu\ c} \equiv f^a{}_{bc}\, \omega_\mu^b = -E_c^\nu\, \nabla_\mu\, e_\nu^a - \frac{1}{2}\, E_c^{\nu\lambda}\, \nabla_\mu\, e^a_{\nu\lambda} \tag{4.28}$$

Then use of (A.7) yield

$$\omega_\mu^a = \omega_\mu^a(e) \equiv \frac{1}{12}\; f^{ab}{}_c\; E_b^\lambda\; \nabla_\mu\; e_\lambda^c + \frac{1}{24}\; f^{ab}{}_c\; E_b^{\lambda\rho}\; \nabla_\mu\; e^c_{\lambda\rho}. \tag{4.29}$$

Into the full covariant derivatives for e_μ^a, $e_{\mu\nu}^a$ and ρ^a introduced above the spin connection $\omega_\mu^a(e)$ (4.29) is to be substituted.

4.4 The second-order action

Now we substitute the solution (4.29) into the action (2.4) and obtain the second-order action.

$$S_{\text{ 2nd order}} = \frac{k}{\pi}\; \int \mathrm{tr}\; e \wedge \left(d\omega(e) + \omega(e) \wedge \omega(e) + \frac{1}{3}\; e \wedge e \right) \tag{4.30}$$

Here we do not consider the boundary terms. To derive classical eqs of motion from this action, $\omega_\mu^a(e)$ is to be varied as a functional of e_μ^a.

In sec. 7 we will derive the generalized Riemann curvature tensor $R^M{}_{N\nu\lambda}$, (6.11). The

above action in the second-order formalism can be reexpressed in terms of this tensor.

$$
\begin{aligned}
S_{\text{2nd order}} &= \frac{k}{12\pi} \int d^3x \left\{ -\epsilon^{\mu\nu\lambda} \left(f^a{}_{bc}\, e^c_\mu\, e^b_M\, E^N_a \right) R^M{}_{N\nu\lambda} + 4\, \epsilon^{\mu\nu\lambda}\, f^a{}_{bc}\, e^a_\mu\, e^b_\nu\, e^c_\lambda \right\} \\
&= \frac{k}{12\pi} \int d^3x \left\{ -\epsilon^{\mu\nu\lambda}\, f^a{}_{bc}\, e^c_\mu \left(e^b_\rho\, E^\sigma_a\, R^\rho{}_{\sigma\nu\lambda} + \frac{1}{2}\, e^b_{\rho\zeta}\, E^\sigma_a\, R^{(\rho\zeta)}{}_{\sigma\nu\lambda} \right.\right. \\
&\quad \left.\left. + \frac{1}{2}\, e^b_\rho\, E^{\sigma\kappa}_a\, R^\rho{}_{(\sigma\kappa)\nu\lambda} + \frac{1}{4}\, e^b_{\rho\zeta}\, E^{\sigma\kappa}_a\, R^{(\rho\zeta)}{}_{(\sigma\kappa)\nu\lambda} \right) + 4\, \epsilon^{\mu\nu\lambda}\, f_{abc}\, e^a_\mu\, e^b_\nu\, e^c_\lambda \right\}
\end{aligned}
\tag{4.31}
$$

See sec.7 for the derivation. Here, $f^a{}_{bc}\, e^c_\mu\, e^b_M\, E^N_a$ is a metric-like quantity which is insensitive to the local frame rotations, while $\epsilon^{\mu\nu\lambda}\, f_{abc}\, e^a_\mu\, e^b_\nu\, e^c_\lambda$ is the generalized cosmological term. So the action integral is expressed in terms of the connections $\Gamma^N_{\mu M}$ and the metric-like fields.

Under the local frame rotations the vielbein transforms as (2.6), and it is easy to show that the spin connection $\omega^a_\mu(e)$ transforms as (2.7). Next, under the generalized diffeomorphisms, the vielbein transforms as (2.8). This can be rewritten as

$$
\delta_1\, e^a_\mu = D_\mu\, \Lambda^a_- = D_\mu\, (\tilde{\xi}^M\, e^a_M) = D_\mu\, (\tilde{\xi}^\nu\, e^a_\nu + \frac{1}{2}\, \tilde{\xi}^{(\nu\lambda)}\, e^a_{\nu\lambda}) \tag{4.32}
$$

Here $\tilde{\xi}^M$ represents for $\tilde{\xi}^\nu$ and $\tilde{\xi}^{\nu\lambda}$, and they are defined by

$$
\tilde{\xi}^M = \Lambda^a_-\, E^M_a = G^{MN}\, \xi_N. \tag{4.33}
$$

These are the local parameters of the generalised diffeomorphisms. For the definition of G^{MN} and the notation $M = \mu, (\mu\nu)$, see appendix E. The tilde in the notation $\tilde{\xi}^M$ means that the metric tensor G^{MN} is used to raise the indices. Because e^a_μ and $e^a_{\mu\nu}$ are covariantly constant, we have

$$
\delta_1\, e^a_\mu = e^a_\nu\, \nabla_\mu\, \tilde{\xi}^\nu + \frac{1}{2}\, e^a_{\nu\lambda}\, \nabla_\mu\, \tilde{\xi}^{(\nu\lambda)}. \tag{4.34}
$$

This does not look like a transformation rule for a covariant vector. However, one can perform, additionally, a local frame rotation $\delta_2\, e^a_\mu = f^a{}_{bc}\, e^b_\mu\, \Lambda^c_+$ with $\Lambda^a_+ = \omega^a_\nu(e)\, \tilde{\xi}^\nu$. The combined transformation is

$$
\delta_{\text{diffeo}}\, e^a_\mu = \delta_1\, e^a_\mu + \delta_2\, e^a_\mu = e^a_\nu\, \nabla_\mu\, \tilde{\xi}^\nu + \tilde{\xi}^\nu\, \nabla_\nu\, e^a_\mu + \frac{1}{2}\, e^a_{\nu\lambda}\, \nabla_\mu\, \tilde{\xi}^{(\nu\lambda)}. \tag{4.35}
$$

Here the torsion-free condition is used to replace $f^a{}_{bc}\, \omega^b_\nu(e)\, e^c_\mu$ by $-\nabla_\nu\, e^a_\mu$. This is the generalization of the diffeomorphism for the vielbein to the spin-3 gravity theory. If $\tilde{\xi}^{(\nu\lambda)} = 0$, the above eq agrees with the transformation rule of a covariant vector.

The new transformation rule of $\omega^a_\mu(e)$ can also be obtained by combining the generalized diffeomorphism $\delta_1\, \omega^a_\mu(e) = f^a{}_{bc}\, e^b_\mu\, \Lambda^c_-$ with the local frame rotation $\delta_2\, \omega^a_\mu(e) = \partial_\mu\, \Lambda^a_+ +$

$f^a{}_{bc}\,\omega^b_\mu(e)\,\Lambda^c_+$.

$$\begin{aligned}\delta_{\rm diffeo}\,\omega^a_\mu(e) &= \delta_1\,\omega^a_\mu(e)+\delta_2\,\omega^a_\mu(e)\\ &= \omega^a_\nu(e)\,\nabla_\mu\,\tilde\xi^\nu+\tilde\xi^\nu\,\nabla_\nu\,\omega^a_\mu(e)-\frac{1}{2}\,\tilde\xi^{\lambda\rho}\,\Gamma^\nu_{\mu,(\lambda\rho)}\,\omega^a_\nu(e)+\frac{1}{2}\,\tilde\xi^{\nu\lambda}\,f^a{}_{bc}\,e^b_\mu\,e^c_{\nu\lambda}.\end{aligned} \tag{4.36}$$

Here in computing $\nabla_\nu\,\omega^a_\mu$ we must set $\omega^a_{\mu\nu}=0$, since this extra component does not exist.[10] On the righthand side of (4.36) a term $\tilde\xi^\nu\,(R^a{}_{\mu\nu}+f^a{}_{bc}\,e^b_\mu\,e^c_\nu)$ is actually present, but this is dropped here, since this term vanishes when the eq of motion is used. In the spin-2 case the transformation rule obtained by dropping the eq of motion term coincides with the diffeomorphism of the spin connection in the second-order formalism.[31] We also expect that (4.36) without the eq of motion term is similarly true in the second-order formalism, since the spin-3 gravity contains the spin-2 gravity. Anyway, the transformation rule of $\omega^a_\mu(e)$ must be checked explicitly by using the definition (4.29) and the expressions for Γ's. We will not attempt to directly prove (4.36) in this paper. Whatever the transformation rule of $\omega^a_\mu(e)$ is, it is possible to show that the action integral in the second-order formalism, (4.30), is invariant under (4.35) in the bulk. This is because when computing the variation of the action integral, $\delta\,\omega^a_\mu(e)$ is multiplied by the torsion (3.2), which vanishes.

5 Generalized diffeomorphism for $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$

The CS theory (2.4) has generalized diffeomorphism invariance (2.9). In this section the transformation rules of the gauge fields $g_{\mu\nu}$ and $\phi_{\mu\nu\lambda}$ will be derived.

5.1 Transformation of $g_{\mu\nu}$

Let us first consider the metric field (2.10). This transforms as

$$\begin{aligned}\delta g_{\mu\nu} &= \frac{1}{2}\,{\rm tr}\,\left(\partial_\mu\,\Lambda+[\omega_\mu,\,\Lambda]\right)\,e_\nu+(\mu\leftrightarrow\nu)\\ &= \frac{1}{2}\,\partial_\mu\,{\rm tr}\,\Lambda\,e_\nu-\frac{1}{2}\,{\rm tr}\,\Lambda\,(\partial_\mu\,e_\nu+[\omega_\mu,\,e_\nu])+(\mu\leftrightarrow\nu)\end{aligned} \tag{5.1}$$

For simplicity of notation, Λ_- in (2.9) is here denoted as Λ. Now, two variation functions are introduced.

$$\xi_\mu=\frac{1}{2}\,{\rm tr}\,\Lambda\,e_\mu, \tag{5.2}$$

$$\zeta_{\mu\nu}+g_{(\mu\nu)\lambda}\,\xi^\lambda=\xi_{(\mu\nu)}=\frac{1}{2}\,{\rm tr}\,\Lambda\,e_{\mu\nu} \tag{5.3}$$

[10]For the discussion of the extra components $\omega^a_{\mu\nu}$ and $\nabla_{(\mu\nu)}$, see the comment at the end of sec.6.

ξ_μ is the coordinate variation and $\zeta_{\mu\nu}$ the spin-3 gauge parameter. In the second eq an extra parameter $\xi_{(\mu\nu)}$ is also introduced. With the help of (3.10), $\delta g_{\mu\nu}$ can be put into the form.

$$\begin{aligned}\delta g_{\mu\nu} &= \partial_\mu \xi_\nu - \Gamma^\lambda_{\mu\nu}\,\xi_\lambda - \frac{1}{2}\,\Gamma^{\lambda\rho}_{\mu\nu}\,(\zeta_{\lambda\rho} + g_{(\lambda\rho)\sigma}\,\xi^\sigma) + (\mu \leftrightarrow \nu) \\ &= \hat{\nabla}_\mu\,\xi_\nu + \hat{\nabla}_\nu\,\xi_\mu - \Gamma^{\lambda\rho}_{\mu\nu}\,\zeta_{\lambda\rho} \end{aligned} \tag{5.4}$$

Here (3.17) is used and $\hat{\nabla}_\mu$ is the ordinary covariant derivative that uses Christoffel symbol $\hat{\Gamma}^\lambda_{\mu\nu}$. Therefore those parts which depend on ξ_μ are the ordinary diffeomorphism. The remaining term, which depends on $\zeta_{\mu\nu}$, is the new spin-3 gauge transformation. This term depends non-linearly on gauge fields such as $g_{(\mu\nu)(\lambda\rho)}$, via $J^{(\mu\nu)(\kappa\sigma)}$, since $\Gamma^{\lambda\rho}_{\mu\nu}$ does also.

Interestingly, this infinitesimal transformation can also be written as

$$\delta\, g_{\mu\nu} = \nabla_\mu\,\xi_\nu + \nabla_\nu\,\xi_\mu \tag{5.5}$$

by adopting the covariant derivative (4.23) introduced in sec.4. Here $\xi_{(\mu\nu)}$ is used as the partner of ξ_μ in computing the derivative. The above result can also be derived by using $g_{\mu\nu} = e^a_\mu\, e_{a\nu}$ and the transformation rule of e^a_μ.

5.2 Transformation of $\phi_{\mu\nu\lambda}$

Let us next turn to the spin-3 gauge field $\phi_{\mu\nu\lambda}$. In this case the variation can be rewritten as follows.

$$\begin{aligned}\delta\,\phi_{\mu\nu\lambda} &= \frac{1}{4}\,\mathrm{tr}\;(\partial_\mu\,\Lambda + [\omega_\mu,\,\Lambda])\;\{e_\nu,\,e_\lambda\} \qquad + (2 \text{ cyclic permutations of } \;\mu,\nu,\lambda) \\ &= \frac{1}{2}\,\partial_\mu\,\mathrm{tr}\,\Lambda\,e_{\nu\lambda} + \frac{1}{6}\,\partial_\mu\,(g_{\nu\lambda}\,\mathrm{tr}\,\rho\Lambda) - \frac{1}{2}\,\Gamma^\rho_{\mu\nu}\,\mathrm{tr}\,\Lambda\,e_{\rho\lambda} - \frac{1}{6}\,\Gamma^\rho_{\mu\nu}\,g_{\rho\lambda}\,\mathrm{tr}\,\Lambda\,\rho \\ &\quad -\frac{1}{2}\,\Gamma^\rho_{\mu\lambda}\,\mathrm{tr}\,\Lambda\,e_{\rho\nu} - \frac{1}{6}\,\Gamma^\rho_{\mu\lambda}\,g_{\rho\nu}\,\mathrm{tr}\,\Lambda\,\rho - \frac{1}{8}\,\Gamma^{\rho\sigma}_{\mu\nu}\,\mathrm{tr}\,\Lambda\,\{e_{\rho\sigma}, e_\lambda\} \\ &\quad -\frac{1}{8}\,\Gamma^{\rho\sigma}_{\mu\lambda}\,\mathrm{tr}\,\Lambda\,\{e_{\rho\sigma}, e_\nu\} + (\text{permutations}) \end{aligned} \tag{5.6}$$

Here (3.5) and (4.4) are used.

In this expression $\mathrm{tr}\,\Lambda\,e_{\nu\lambda}$ is rewritten by means of (5.3). To compute other terms involving Λ, (5.2) and (5.3) must be solved for $\Lambda = t_a\,\Lambda^a$. Multiplying $\xi_\mu = \Lambda_a\,e^a_\mu$ and $\zeta_{\mu\nu} + g_{(\mu\nu)\lambda}\,\xi^\lambda = \Lambda_a\,e^a_{\mu\nu}$ by E^μ_b and $\frac{1}{2}\,E^{\mu\nu}_b$, respectively, and adding the two, the following formula is derived.

$$\Lambda_a = \xi_\mu\,E^\mu_a + \frac{1}{2}\,\left(\zeta_{\mu\nu} + g_{(\mu\nu)\lambda}\,\xi^\lambda\right)\,E^{\mu\nu}_a = \xi_\mu\,E^\mu_a + \frac{1}{2}\,\xi_{(\mu\nu)}\,E^{\mu\nu}_a \tag{5.7}$$

The trace $\mathrm{tr}\,\Lambda\rho$ is then reexpressed as follows.

$$\begin{aligned}&\mathrm{tr}\,\Lambda\rho = 2\,\Lambda_a\,\rho^a = \zeta_{\rho\sigma}\,E^{\rho\sigma}_a\,\rho^a + 2\,\xi^\sigma\,(g_{\sigma\lambda}\,E^\lambda_a + \frac{1}{2}\,g_{(\rho\kappa)\sigma}\,E^{\rho\kappa}_a)\,\rho^a \\ &= \zeta_{\rho\sigma}\,E^{\rho\sigma}_a\,\rho^a + \xi^\sigma\,\rho^a\,\mathrm{tr}\,e_\sigma\,t_a = \zeta_{\rho\sigma}\,E^{\rho\sigma}_a\,\rho^a + \xi^\sigma\;\mathrm{tr}\;e_\sigma\,\rho \end{aligned} \tag{5.8}$$

Here formula (C.2) for t_a given in appendix C is used. The term $\mathrm{tr}\,\Lambda\,\{e_{\rho\sigma},\,e_\lambda\}$ in (5.6) is similarly computed as follows.

$$
\begin{aligned}
\mathrm{tr}\,\Lambda\,\{e_{\rho\sigma},\,e_\lambda\} &= 2\,\xi_\mu\,d^a{}_{bc}\,E^\mu_a\,e^b_{\rho\sigma}\,e^c_\lambda + \xi^\alpha\,g_{(\mu\nu)\alpha}\,d^a{}_{bc}\,e^b_{\rho\sigma}\,e^c_\lambda\,E^{\mu\nu}_a \\
&\quad +\zeta_{\mu\nu}\,d^a{}_{bc}\,E^{\mu\nu}_a\,e^b_{\rho\sigma}\,e^c_\lambda
\end{aligned}
\tag{5.9}
$$

The variation $\delta\phi_{\mu\nu\lambda}$ will be decomposed into $\delta_\xi\,\phi_{\mu\nu\lambda} + \delta_\zeta\,\phi_{\mu\nu\lambda}$. Let us first concentrate on those terms which depend on ξ. After some calculation one obtains

$$
\begin{aligned}
\delta_\xi\,\phi_{\mu\nu\lambda} &= \hat{\nabla}_\mu\,(\xi^\sigma\,\phi_{\nu\lambda\sigma}) - \frac{1}{2}\,\xi^\alpha\,\Gamma^{\sigma\kappa}_{\mu\nu}\,M_{(\sigma\kappa)(\lambda\alpha)} - \frac{1}{2}\,\xi^\alpha\,\Gamma^{\sigma\kappa}_{\mu\lambda}\,M_{(\sigma\kappa)(\nu\alpha)} \\
&\quad -\frac{1}{12}\,\Gamma^{\sigma\kappa}_{\mu\nu}\,\xi_\lambda\,W_{\sigma\kappa} - \frac{1}{12}\,\Gamma^{\sigma\kappa}_{\mu\lambda}\,\xi_\nu\,W_{\sigma\kappa} + (\text{cyclic permutations})
\end{aligned}
\tag{5.10}
$$

The last two terms which contain $W_{\sigma\kappa}$ can be rewritten using (3.29) as

$$
-\frac{1}{12}\,\Gamma^{\sigma\kappa}_{\mu\nu}\,\xi_\lambda\,W_{\sigma\kappa} = -\frac{1}{5}\,\xi_\lambda\,\hat{\nabla}_\mu\,\phi_{\nu\alpha}{}^\alpha + \frac{1}{9}\,\xi_\lambda\,\hat{\nabla}^\alpha\,\Phi_{\mu\nu\alpha} + g^{\alpha\beta}\,S_{\mu\alpha,\nu\beta}.
\tag{5.11}
$$

Here $\Phi_{\mu\nu\alpha}$ (3.27) is the traceless part of $\phi_{\mu\nu\alpha}$. Those terms which contain $M_{(\sigma\kappa)(\lambda\alpha)}$ can be rewritten by using (3.29), as

$$
\begin{aligned}
-\frac{1}{2}\,\xi^\alpha\,\Gamma^{\sigma\kappa}_{\mu\nu}\,M_{(\sigma\kappa)(\lambda\alpha)} &= -\frac{1}{3}\,\xi^\alpha\,\left(\hat{\nabla}_\mu\,\Phi_{\nu\lambda\alpha} + \hat{\nabla}_\nu\,\Phi_{\mu\lambda\alpha}\right) + \frac{1}{6}\,\xi^\alpha\,\left(\hat{\nabla}_\lambda\,\Phi_{\mu\nu\alpha} + \hat{\nabla}_\alpha\,\Phi_{\mu\nu\lambda}\right) \\
&\quad -\frac{1}{9}\,\xi_\lambda\,\hat{\nabla}^\kappa\,\Phi_{\mu\nu\kappa} - \frac{1}{2}\,\xi^\alpha\,S_{\mu\nu,\lambda\alpha}.
\end{aligned}
\tag{5.12}
$$

Finally, the variation $\delta_\xi\,\phi_{\mu\nu\lambda}$ is given by

$$
\begin{aligned}
\delta_\xi\,\phi_{\mu\nu\lambda} &= \xi^\sigma\,\hat{\nabla}_\sigma\,\phi_{\mu\nu\lambda} + \hat{\nabla}_\mu\,\xi^\sigma\,\phi_{\sigma\nu\lambda} + \hat{\nabla}_\lambda\,\xi^\sigma\,\phi_{\sigma\nu\mu} + \hat{\nabla}_\nu\,\xi^\sigma\,\phi_{\sigma\mu\lambda} \\
&\quad +\frac{1}{5}\,g_{\mu\nu}\,\xi^\sigma\,\left(\hat{\nabla}_\lambda\,\phi_{\sigma\kappa}{}^\kappa - \hat{\nabla}_\sigma\,\phi_{\lambda\kappa}{}^\kappa\right) + \frac{1}{5}\,g_{\nu\lambda}\,\xi^\sigma\,\left(\hat{\nabla}_\mu\,\phi_{\sigma\kappa}{}^\kappa - \hat{\nabla}_\sigma\,\phi_{\mu\kappa}{}^\kappa\right) \\
&\quad +\frac{1}{5}\,g_{\mu\lambda}\,\xi^\sigma\,\left(\hat{\nabla}_\nu\,\phi_{\sigma\kappa}{}^\kappa - \hat{\nabla}_\sigma\,\phi_{\nu\kappa}{}^\kappa\right) \\
&\quad + \left\{g^{\alpha\beta}\,(S_{\mu\alpha,\nu\beta}\,\xi_\nu + S_{\mu\alpha,\lambda\beta}\,\xi_\nu) - \xi^\alpha\,S_{\mu\nu,\lambda\alpha} + \text{cyclic permutations of } \mu,\nu,\lambda\right\}.
\end{aligned}
\tag{5.13}
$$

Therefore except for the trace parts and the terms containing $S_{\mu\nu,\lambda\rho}$, the spin-3 gauge field $\phi_{\mu\nu\lambda}$ transforms as a spin-3 tensor under ordinary diffeomorphisms (ξ_μ).

Those terms which depend on $\zeta_{\mu\nu}$ can also be worked out. After certain amount of

calculation the ζ transformation of the spin-3 gauge field is obtained.

$$\begin{aligned}
\delta_\zeta \phi_{\mu\nu\lambda} \;\; = \;\; & \hat{\nabla}_\mu \, \zeta_{\nu\lambda} + \frac{1}{6}\, g_{\nu\lambda}\, \hat{\nabla}_\mu \, (\zeta_{\rho\sigma}\, E^{\rho\sigma}_a \, \rho^a) + \frac{1}{12}\, (\zeta_{\alpha\beta}\, E^{\alpha\beta}_a\, \rho^a)\, (\Gamma^{\sigma\kappa}_{\mu\nu}\, \phi_{\sigma\kappa\lambda} + \Gamma^{\sigma\kappa}_{\mu\lambda}\, \phi_{\sigma\kappa\nu}) \\
& + \frac{1}{2}\, \phi_{\sigma\kappa}{}^{\rho}\, (\Gamma^{\sigma\kappa}_{\mu\nu}\, \zeta_{\rho\lambda} + \Gamma^{\sigma\kappa}_{\mu\lambda}\, \zeta_{\rho\nu}) \\
& + \frac{1}{8}\, (\Gamma^{\kappa\sigma}_{\mu\nu}\, g_{(\kappa\sigma)(\lambda\rho)} + \Gamma^{\kappa\sigma}_{\mu\lambda}\, g_{(\kappa\sigma)(\nu\rho)})\, \phi^{\rho}{}_{(\tau\eta)}\, J^{(\tau\eta)(\alpha\beta)}\, \zeta_{\alpha\beta} \\
& - \frac{1}{16}\, (\Gamma^{\kappa\sigma}_{\mu\nu}\, g_{(\kappa\sigma)(\rho\tau)\lambda} + \Gamma^{\kappa\sigma}_{\mu\lambda}\, g_{(\kappa\sigma)(\rho\tau)\nu})\, J^{(\rho\tau)(\alpha\beta)}\, \zeta_{\alpha\beta} \\
& + \frac{1}{24}\, (\Gamma^{\kappa\sigma}_{\mu\nu}\, \phi_{\lambda\tau\eta} + \Gamma^{\kappa\sigma}_{\mu\lambda}\, \phi_{\nu\tau\eta})\, \phi_{(\kappa\sigma)(\rho\gamma)}\, g^{\rho\gamma}\, J^{(\tau\eta)(\alpha\beta)}\, \zeta_{\alpha\beta} \\
& + (\text{2 cyclic permutations})
\end{aligned} \tag{5.14}$$

Here those terms which include ρ can be simplified further by using (4.14)-(4.15). The gauge fields $g_{(\kappa\sigma)(\rho\tau)\lambda}$, $g_{(\kappa\sigma)(\nu\rho)}$ and $\phi_{(\kappa\sigma)(\rho\gamma)}$ are defined in appendix C. Under a new spin-3 gauge transformation ($\zeta_{\mu\nu}$), $\phi_{\mu\nu\lambda}$ transforms in a complicated way which depends on higher-indexed gauge fields. Transformations of these gauge fields must also be studied. However, in this paper this will not be attempted.

At the beginning of this section it was shown that by using the new covariant derivative ∇_μ, the transformation $\delta g_{\mu\nu}$ can be compactly expressed as (5.5) just like in Einstein gravity. Then one may expect that due to relations among gauge fields, the transformation $\delta\phi_{\mu\nu\lambda} = \delta_\xi \phi_{\mu\nu\lambda} + \delta_\zeta \phi_{\mu\nu\lambda}$ might also be succinctly written.

Actually, using $\delta\, e^a_\mu = D_\mu\, \Lambda^a$ and $\delta\, e^a_{\mu\nu} = \frac{1}{2}\, d^a{}_{bc}\, (e^b_\mu\, D_\nu\, \Lambda^c + e^b_\nu\, D_\mu\, \Lambda^c) - \frac{1}{3}\, \rho^a\, (\nabla_\mu\, \xi_\nu + \nabla_\nu\, \xi_\mu) - \frac{1}{3}\, g_{\mu\nu}\, \delta\rho^a$, one can show that

$$\begin{aligned}
\delta\, \phi_{\mu\nu\lambda} \;\; = \;\; & \nabla_\mu \left(\xi_{(\nu\lambda)} + \frac{1}{3}\, \rho^a\, \Lambda_a\, g_{\nu\lambda} \right) + \nabla_\nu \left(\xi_{(\lambda\mu)} + \frac{1}{3}\, \rho^a\, \Lambda_a\, g_{\lambda\mu} \right) \\
& + \nabla_\lambda \left(\xi_{(\mu\nu)} + \frac{1}{3}\, \rho^a\, \Lambda_a\, g_{\mu\nu} \right).
\end{aligned} \tag{5.15}$$

Except for the trace terms this agrees with the expected transformation rule of the spin-3 gauge field.

6 Parallel transport and Curvature tensor

To investigate the spin-3 geometry, it is useful to introduce a parallel transport matrix, holonomy matrix and curvature tensor. This will be done in this section.

Let $v^a(x)$ be a vector field in the local Lorentz frame. For an arbitrary curve $x^\mu = x^\mu(s)$, this vector is said to be parallel transported along the curve,[31] if it satisfies the equation

$$\frac{dv^a}{ds} + \omega_\mu{}^a{}_b(e)\, \frac{dx^\mu}{ds}\, v^b = 0. \tag{6.1}$$

This equation can be solved in terms of the ordered exponential

$$\begin{aligned} v^a(x(s)) &= U^a{}_b(s,0)\, v^b(x(0)), && (6.2)\\ U^a{}_b(s,0) &= \left(P\,\exp\left\{-\int_0^s \omega_\mu \frac{dx^\mu}{ds'}\, ds'\right\}\right)^a{}_b. && (6.3)\end{aligned}$$

Here, as usual, the symbol P denotes path ordering.

$$P(A(s_1)\, B(s_2)) = \begin{cases} A(s_1)\, B(s_2) & (\text{if} \;\; s_1 > s_2), \\ B(s_2)\, A(s_1) & (\text{if} \;\; s_2 > s_1). \end{cases} \tag{6.4}$$

These relations can be converted into that for spacetime quatities by means of the vielbeins. Firstly, we perform the following GL(8,R) gauge transformation on the spin connection matrix $\omega_\mu(e)$.[11]

$$\omega_\mu{}^a{}_b(e) \to \Upsilon_\mu{}^M{}_N = E^M_a\, \omega_\mu{}^a{}_b(e)\, e^b_N + E^M_a\, \partial_\mu\, e^a_N \tag{6.5}$$

Here $M, N = \mu, (\mu\nu)$ are the indices explained in appendix E. By (4.28) the new spin connection can be written as

$$\Upsilon_\mu{}^M{}_N = -E^M_a\, \nabla_\mu\, e^a_N + E^M_a\, \partial_\mu\, e^a_N. \tag{6.6}$$

This agrees with the connections defined in sec.4,

$$\begin{aligned} \Upsilon_\mu{}^\lambda{}_\nu &= \Gamma^\lambda_{\mu\nu}, \qquad \Upsilon_\mu{}^{(\lambda\rho)}{}_\nu = \Gamma^{\lambda\rho}_{\mu\nu}, \\ \Upsilon_\mu{}^\rho{}_{(\nu\lambda)} &= \Gamma^\rho_{\mu,(\nu\lambda)}, \qquad \Upsilon_\mu{}^{(\rho\sigma)}{}_{(\nu\lambda)} = \Gamma^{(\rho\sigma)}_{\mu,(\nu\lambda)}. \end{aligned} \tag{6.7}$$

Under the gauge transformation (6.5) the path-ordered exponential (6.3) transforms into a spacetime quantity,

$$U^M{}_N(s,0) = \left(P\,\exp\left\{-\int_0^s \Upsilon_\mu \frac{dx^\mu}{ds'}\, ds'\right\}\right)^M{}_N. \tag{6.8}$$

The parallel transport eq (6.1) is also rewritten as

$$\frac{dv^M}{ds} + \Upsilon_\mu{}^M{}_N \frac{dx^\mu}{ds}\, v^N = 0. \tag{6.9}$$

If the curve $x^\mu(s), (0 \leq s \leq 1)$ is closed, the matrix $U^M_N(1,0)$ defines a holonomy matrix. For an infinitesimal closed curve γ which encloses a small surface S, this holonomy can be evaluated by expansion of the exponential. By using Stokes's theorem this yields a generalization of the Riemann curvature tensor at the lowest order of expansion.

$$U^M{}_N(1,0) = \delta^M{}_N + \int_S R^M{}_{N\mu\nu}\, d\Sigma^{\mu\nu} + \ldots \tag{6.10}$$

[11] In the case of Einstein gravity a similar transformation is used.[30]

Here

$$R^M{}_{N\mu\nu} \equiv \partial_\mu \Gamma_\nu{}^M{}_N - \partial_\nu \Gamma_\mu{}^M{}_N - \Gamma_\nu{}^M{}_K \Gamma_\mu{}^K{}_N + \Gamma_\mu{}^M{}_K \Gamma_\nu{}^K{}_N. \tag{6.11}$$

The action integral in the second-order formalism (4.30) can be expressed in terms of this curvature tensor. To do this, we perform the $GL(8,R)$ gauge transformation $\omega^a_{\mu b} = e^a_M \Gamma^M_{\mu N} E^N_b - E^N_b \partial_\mu e^a_N$ on the curvature tensor $R^a{}_{b\mu\nu} = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c} \omega^c_{\nu b} - \omega^a_{\nu c} \omega^c_{\mu b}$. Since the curvature 2-form is gauge covariant, we obtain

$$R^a{}_{b\mu\nu} = e^a_M E^N_b R^M{}_{N\mu\nu}. \tag{6.12}$$

The identity $R^a{}_{b\mu\nu} = f^a{}_{cb} R^c_{\mu\nu}$, where $R^c_{\mu\nu} = \partial_\mu \omega^c_\nu - \partial_\nu \omega^c_\mu + f^c{}_{de} \omega^d_\mu \omega^e_\nu$, leads to (4.31).

In the spin-2 gravity theory, the Riemann curvature tensor also defines the commutator of the covariant derivatives, $[\nabla_\nu, \nabla_\mu] v^\lambda = R^\lambda{}_{\rho\nu\mu} v^\rho$. In this theory, this eq can be derived by starting with the curvature 2-form $R^a{}_{b\nu\mu} v^b = [D_\nu, D_\mu] v^a$ and by projecting onto the base space using the vielbein as $v^\mu = v^a E^\mu_a$. In the spin-3 case we also expect similar formulae such as

$$\begin{aligned} [\nabla_\nu, \nabla_\mu] v^\lambda &= R^\lambda{}_{\rho\mu\nu} v^\rho + \frac{1}{2} R^\lambda{}_{(\rho\sigma)\mu\nu} v^{(\rho\sigma)}, & (6.13) \\ [\nabla_\nu, \nabla_\mu] v^{(\lambda\rho)} &= R^{(\lambda\rho)}{}_{\sigma\mu\nu} v^\sigma + \frac{1}{2} R^{(\lambda\rho)}{}_{(\sigma\kappa)\mu\nu} v^{(\sigma\kappa)}. & (6.14) \end{aligned}$$

However, there is an obstacle in deriving such formulae, because the covariant derivative does not have the component '$\nabla_{(\mu\nu)}$' in the new direction $(\mu\nu)$.

If this component exists, it is possible to compute the commutators of the covariant derivatives. Actually, we have $\nabla_\nu \nabla_\mu v^\lambda = \partial_\nu \partial_\mu v^\lambda + (\partial_\nu \Gamma^\lambda_{\mu M}) v^M + \Gamma^\lambda_{\mu M} \partial_\nu v^M - \Gamma^M_{\nu\mu} \nabla_M v^\lambda + \Gamma^\lambda_{\nu M} \partial_\mu v^M + \Gamma^\lambda_{\nu M} \Gamma^M_{\mu N} v^N$ and then $[\nabla_\nu, \nabla_\mu] v^\lambda = (\partial_\nu \Gamma^\lambda_{\mu M} - \partial_\mu \Gamma^\lambda_{\nu M} + \Gamma^\lambda_{\nu N} \Gamma^N_{\mu M} - \Gamma^\lambda_{\mu N} \Gamma^N_{\nu M}) v^M = R^\lambda{}_{M\nu\mu} v^M$. The term $\Gamma^M_{\nu\mu} \nabla_M v^\lambda$ cancels out in the commutator. The actual value of $\nabla_{(\mu\nu)} v^\lambda$ does not matter. It is important to notice that *it can even be zero*: $\nabla_{(\mu\nu)} v^\lambda = 0$.

In order to define $\nabla_{(\mu\nu)}$, then, we would need to introduce new coordinates $x^{\mu\nu}$ and set $\nabla_{(\mu\nu)} v^\lambda = \partial_{(\mu\nu)} v^\lambda + \Gamma^\lambda_{(\mu\nu),M} v^M$. This, however, would make the spacetime have dimension 8, and one would need to cope with a problem of integrating over the new coordinates. So one of possible prescriptions would be to avoid introducing $x^{\mu\nu}$ and set $\nabla_{(\mu\nu)} v^\lambda = \Gamma^\lambda_{(\mu\nu),M} v^M$. We would then also need to introduce a new component of the spin connection, $\omega^a_{\mu\nu}$, and impose a torsion-free condition, $\nabla_{(\mu\nu)} e^a_M + f^a{}_{bc} \omega^b_{\mu\nu} e^c_M = 0$. However, compatibility of this covariant derivative $\nabla_{(\mu\nu)}$ with $g_{\lambda\rho}$ and $g_{(\lambda\rho)\sigma}$ would inevitably lead to the conclusion $\Gamma^N_{(\mu\nu),M} = 0$ and $\omega^a_{\mu\nu} = 0$. To define $\nabla_{(\mu\nu)}$, introduction of extra coordinates seems unavoidable. Therefore, we will set $\nabla_{(\mu\nu)} = 0$ in this paper. Even in this case the rules (6.13)-(6.14) of the commutators of the covariant derivatives still apply.

7 Gravitational CS term

In 3D there also exists a gravitational Chern-Simons term.[28] It is given by

$$S_{\mathrm{GCS}}(\omega) \;=\; \frac{k}{8\pi\mu}\int d^3x\,\epsilon^{\mu\nu\lambda}\left(\omega(e)^a_{\mu b}\,\partial_\nu\,\omega(e)^b_{\lambda a}+\frac{2}{3}\,\omega(e)^a_{\mu b}\,\omega(e)^b_{\nu c}\,\omega(e)^c_{\lambda a}\right). \quad (7.1)$$

Here μ is a constant. In this action, the spin connection $\omega^a_{\mu b}(e)$ is a functional of the vielbein e^a_μ, $e^a_{\mu\nu}$ as defined in (4.29). This action is invariant in the bulk under both the local frame transformation and the generalized diffeomorphism. The invariance is broken at the boundary. If this term is added to the CS action in the second-order formalism (4.31), the action of a topological massive spin-3 gravity (a generalization of the topological massive gravity[28]) is obtained. In the gravity/CFT correspondence the gravitational action with the gravitational CS term corresponds to a left-right asymmetric (chiral) CFT. This action has derivatives of cubic order and hence the eqs of motion will contain terms with cubic-order derivatives. Let us note that if the solution for the spin connection is not substituted into the action integral, and the vielbein and the spin connection were treated independently, the torsion-free eq would be modified. In order to avoid this, the torsion-free condition may be imposed by means of a Lagrange multiplier field.[29][12] However, the generalized diffeomorphism invariance (2.9) will be broken by the multiplier term.[12]

It is known that in the case of ordinary 3D spin-2 gravity, the gravitational Chern-Simons term can also be expressed in terms of the Christoffel connection up to a winding number term; $S_{\mathrm{GCS}}(\omega) = S^{\text{spin-2}}_{\mathrm{GCS}}(\hat{\Gamma})+$ (winding number term).[28][30]

$$S^{\text{spin-2}}_{\mathrm{GCS}}(\hat{\Gamma}) = \frac{k}{8\pi\mu}\int d^3x\,\epsilon^{\mu\nu\lambda}\,(\hat{\Gamma}^\rho_{\mu\sigma}\,\partial_\nu\,\hat{\Gamma}^\sigma_{\lambda\rho}+\frac{2}{3}\,\hat{\Gamma}^\sigma_{\mu\kappa}\,\hat{\Gamma}^\kappa_{\nu\rho}\,\hat{\Gamma}^\rho_{\lambda\sigma}) \quad (7.2)$$

Actually, this last form of the gravitational CS term must be used in the second-order formalism. In the case of spin-3 gravity, a similar expression for the action can be derived by using the gauge transformation (6.5). After substitution we have, up to winding number terms,

$$\begin{aligned}
S^{\text{spin-3}}_{\mathrm{GCS}}(\Gamma) \;&=\; \frac{k}{8\pi\mu}\int d^3x\,\epsilon^{\mu\nu\lambda}\left(\Upsilon_\mu{}^M{}_N\,\partial_\nu\,\Upsilon_\lambda{}^N{}_M+\frac{2}{3}\,\Upsilon_\mu{}^M{}_N\,\Upsilon_\nu{}^N{}_K\,\Upsilon_\lambda{}^K{}_M\right)\\
&=\; \frac{k}{8\pi\mu}\int d^3x\,\epsilon^{\mu\nu\lambda}\left(\Gamma^\rho_{\mu\sigma}\partial_\nu\,\Gamma^\sigma_{\lambda\rho}+\frac{1}{2}\,\Gamma^{\rho\sigma}_{\mu\kappa}\,\partial_\nu\,\Gamma^\kappa_{\lambda,(\rho\sigma)}+\frac{1}{2}\,\Gamma^\kappa_{(\rho\sigma)}\,\partial_\nu\,\Gamma^{\rho\sigma}_{\lambda\kappa}\right.\\
&\quad+\frac{1}{4}\,\Gamma^{\rho\sigma}_{\mu,(\kappa\tau)}\,\partial_\nu\,\Gamma^{\kappa\tau}_{\lambda,(\rho\sigma)}+\frac{2}{3}\,\Gamma^\rho_{\mu\sigma}\,\Gamma^\sigma_{\nu\kappa}\,\Gamma^\kappa_{\lambda\rho}+\Gamma^{\rho\tau}_{\mu,\sigma}\,\Gamma^\sigma_{\nu\kappa}\,\Gamma^\kappa_{\lambda,(\rho\tau)}\\
&\quad\left.+\frac{1}{2}\,\Gamma^{\rho\tau}_{\mu,(\sigma\eta)}\,\Gamma^{\sigma\eta}_{\nu\kappa}\,\Gamma^\kappa_{\lambda,(\rho\tau)}+\frac{1}{12}\,\Gamma^{\rho\tau}_{\mu,(\sigma\eta)}\,\Gamma^{\sigma\eta}_{\nu(\kappa\alpha)}\,\Gamma^{\kappa\alpha}_{\lambda,(\rho\tau)}\right). \quad (7.3)
\end{aligned}$$

[12] A linearized action in the topological massive higher-spin gravity is studied in [32]. Topological massive higher-spin gravity with a multiplier field is studied in [33].

In the spin-2 gravity theory, solutions such as BTZ black hole[34] in the theory without the gravitational CS term are known to be also solutions of the eqs of motion of the topologically massive gravity theory. Therefore the natural questions to ask are: do the solutions in the spin-3 gravity without the gravitational CS term, such as the spin-3 black hole[6], also solve the eqs of motion in the spin-3 topologically massive gravity? If it is the case, how the central charges of the W_3 algebras in the boundary CFT and the value of the entropy will be modified in the presence of the gravitational CS term?

The black hole solution with spin-3 charge is asymptotically AdS_3 with AdS radius $1/2$.[6] Therefore it may be interesting to study the existence of propagating gravitons with this asymptotic boundary condition. These problems are left for the future studies.

8 Summary and discussion

In this paper a second-order formalism of the 3D spin-3 gravity is addressed and it is shown that many of the notions and geometrical quantities of Einstein gravity thoery can be introduced into this theory. Extra vielbeins $e^a_{\mu\nu}$ (3.5) are introduced in order to eliminate the spin connection from the CS formulation of the 3D spin-3 gravity in a way covariant under the local frame rotations. It is shown that new connections $\Gamma^N_{\mu M}$ can be introduced and that a covariant derivative ∇_μ (4.2), (4.10) can be defined. The torsion-free condition is solved for the spin connection ω^a_μ as (4.29) in terms of the generalized vielbein and its inverse. In terms of this solution, the action integral in the second-order formalism (4.30) is presented. Then a generalised Riemann curvature tensor for the spin-3 gravity is defined. The explicit form of the generalized diffeomorphism of the metric $g_{\mu\nu}$ and the spin-3 gauge field $\phi_{\mu\nu\lambda}$ is presented. Finally, the action integral for topologically massive spin-3 gravity is presented explicitly.

In the present paper, the transformation rules of the connections $\Gamma^N_{\mu M}$ under the generalized diffeomorphisms are not studied explicitly. This is because the expression for $S_{\mu\nu,\lambda\rho}$ in $\Gamma^{\lambda\rho}_{\mu\nu}$ is complicated. This problem must be studied in the future. However, by assuming the transformation rule of $\omega^a_\mu(e)$ as (4.36) and using the relation between Γ's and ω^a_μ it is possible to derive the transformation rule of Γ's.

For other future work we would like to consider the coupling of matter fields to the spin-3 gravity. For this purpose it is necessary to define density and tensors which transform appropriately under the generalized diffeomorphisms. Then it must be shown that the covariant derivatives of the general tensors also transform as tensors. At present, this remains an unsolved problem.

Finally, there will be several directions for future investigations. To enumerate a few, the geometry of the 3D spin-3 gravity is still not well-understood. This must be studied further and the spin-3 gravity must be formulated from scratch without relying on CS theory. In the case of supergravity, where gravity theory is likewise extended by supersymmetry transformations, one can understand the theory geometrically by introducing supercoordinates, a superspace and superfields. Likewise, it might be possible to better understand the spin-3 gravity analogously by introducing a 'spin-3 space'.

A generalization of the work in this paper to spin-N(≥ 4) gravity theories will be straightforward. For example, in the spin-4 gravity theory, extended vielbeins e_μ, $e_{\mu\nu}$ and $e_{\mu\nu\lambda}$, which are completely symmetric in the indices and satisfy traceless conditions, will provide 3+5+7=15 basis vectors. This number agrees with the dimension of $sl(4, R)$. The case of spin-N gravity works similarly.

A sl(3,R) algebra

Let the generators L_i $(i = -1, 0, 1)$, W_n $(n = -2, \ldots 2)$ satisfy an $sl(3, R)$ algebra.

$$
\begin{aligned}
[L_i, L_j] &= (i-j)\, L_{i+j}, \qquad [L_i, W_n] = (2i - n)\, W_{i+n}, \\
[W_m, W_n] &= -\frac{1}{3}(m-n)\,\{2m^2 + 2n^2 - mn - 8\}\, L_{m+n}
\end{aligned}
\tag{A.1}
$$

We use the same three-dimensional representation as in [2] with the parameter $\sigma = -1$.

$$
\begin{aligned}
L_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \\
W_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad W_0 = \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
W_{-1} &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_{-2} = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{A.2}
$$

Nonvanishing norms of these matrices are given by

$$
\mathrm{tr}\,(L_0)^2 = 2, \quad \mathrm{tr}\,(L_{-1}L_1) = -4, \quad \mathrm{tr}\,(W_0)^2 = \frac{8}{3}, \quad \mathrm{tr}\,(W_1W_{-1}) = -4, \quad \mathrm{tr}\,(W_2W_{-2}) = 16 \tag{A.3}
$$

These generators will also be collectively denoted as $t_a, (a = 1, \ldots, 8)$,

$$
\begin{aligned}
t_1 &= L_1, \quad t_2 = L_0, \quad t_3 = L_{-1}, \\
t_4 &= W_2, \quad t_5 = W_1, \quad t_6 = W_0, \quad t_7 = W_{-1}, \quad t_8 = W_{-2}.
\end{aligned}
\tag{A.4}
$$

The structure constants $f_{ab}{}^c$ are defined by

$$
[t_a, t_b] = f_{ab}{}^c\, t_c. \tag{A.5}
$$

The Killing metric h_{ab} for the local frame is defined by

$$
h_{ab} = \frac{1}{2}\,\mathrm{tr}\,(t_a t_b) \tag{A.6}
$$

Its nonzero components are given by $h_{22} = 1,\ h_{13} = h_{31} = -2, h_{48} = h_{84} = 8,\ h_{57} = h_{75} = -2,\ h_{66} = 4/3$. This metric tensor has a signature $(3, 5)$. Indices of the local frame are raised and lowered by h_{ab} and its inverse h^{ab}. Then $f_{abc} \equiv f_{ab}{}^d\, h_{dc}$ is completely anti-symmetric in the three indices. It can be shown that f_{abc} and h_{ab} are related by

$$
h_{ab} = -\frac{1}{12}\, f_a{}^{cd}\, f_{bcd}. \tag{A.7}
$$

The structure constants are given by

$$
\begin{aligned}
&f_{123} = -2,\ f_{158} = 8,\ f_{167} = -4,\ f_{248} = -16, \\
&f_{257} = 2,\ f_{347} = 8,\ f_{356} = -4
\end{aligned}
\tag{A.8}
$$

The invariant tensor ${d_{ab}}^c$ is defined by

$$\{t_a, t_b\} = {d_{ab}}^c\, t_c + {d_{ab}}^0\, t_0, \tag{A.9}$$

where $t_0 = \boldsymbol{I}$ is an identity matrix. The constant with the lowered index $d_{abc} = {d_{ab}}^d\, h_{dc}$ is completely symmetric in all the indices. These constants are given by

$$\begin{aligned} & d_{127} = d_{235} = -2, \quad d_{136} = d_{226} = d_{567} = \frac{4}{3}, \quad d_{118} = d_{334} = 8, \\ & d_{468} = \frac{32}{3}, \quad d_{477} = d_{558} = -8, \quad d_{666} = -\frac{16}{9}, \qquad {d_{ab}}^0 = \frac{4}{3}\, h_{ab} \end{aligned} \tag{A.10}$$

B AdS$_3$

The flat connections which yield AdS_3 spacetime are given by

$$\begin{aligned} A &= e^r\ L_1\, dx^+ + L_0\, dr, \\ \bar{A} &= -e^r\ L_{-1}\, dx^- - L_0\, dr. \end{aligned} \tag{B.1}$$

In this appendix, to avoid confusion of $\mu(=0,1,2)$ with $a(=1,2,...)$ a different notation $\mu = t, \phi, r$ will be used, and $dx^\pm \equiv dt \pm d\phi$. The corresponding vielbein and spin connection are given by

$$\begin{aligned} e_t = \omega_t &= \frac{1}{2}\, e^r\, (L_1 + L_{-1}), \\ e_\phi = \omega_\phi &= \frac{1}{2}\, e^r\, (L_1 - L_{-1}), \\ e_r &= L_0, \qquad \omega_r = 0. \end{aligned} \tag{B.2}$$

The metric is $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = dr^2 + e^{2r}(-dt^2 + d\phi^2)$.

By (B.2) one obtains

$$\begin{aligned} (e_r)^2 &= \frac{1}{2}\ W_0 + \frac{2}{3}\,\boldsymbol{I} && \rightarrow && \hat{e}_{rr} = \frac{1}{2}\ W_0, \\ (e_t)^2 &= \frac{1}{8}\, e^{2r}(W_2 + W_{-2} + 2\, W_0 - \frac{16}{3}\,\boldsymbol{I}) && \rightarrow && \hat{e}_{tt} = \frac{1}{8}\, e^{2r}(W_2 + W_{-2} + 2\, W_0), \\ (e_\phi)^2 &= \frac{1}{8}\ e^{2r}(W_2 + W_{-2} - 2\, W_0 + \frac{16}{3}\,\boldsymbol{I}) && \rightarrow && \hat{e}_{\phi\phi} = \frac{1}{8}\, e^{2r}(W_2 + W_{-2} - 2\, W_0) \end{aligned} \tag{B.3}$$

Since $\hat{e}_{\mu\nu}$ satisfies

$$\rho = g^{\mu\nu}\, \hat{e}_{\mu\nu} = 0, \tag{B.4}$$

one has $e_{\mu\nu} = \hat{e}_{\mu\nu}$ for this special geometry. The other components are given by

$$\begin{aligned} e_{rt} &= \hat{e}_{rt} = \frac{1}{4}\, e^r\, (W_1 + W_{-1}), \\ e_{r\phi} &= \hat{e}_{r\phi} = \frac{1}{4}\, e^r\, (W_1 - W_{-1}), \\ e_{t\phi} &= \hat{e}_{t\phi} = \frac{1}{8}\, e^{2r}\, (W_2 - W_{-2}). \end{aligned} \tag{B.5}$$

The non-vanishing components of the vielbein in terms of the basis t_a are

$$
\begin{aligned}
&e_r^2 = 1,\ e_t^1 = \frac{1}{2}\, e^r,\ e_t^3 = \frac{1}{2}\, e^r,\ e_\phi^1 = \frac{1}{2}\, e^r,\ e_\phi^3 = -\frac{1}{2}\, e^r,\\
&e_{rt}^5 = \frac{1}{4}\, e^r,\ e_{rt}^7 = \frac{1}{4}\, e^r,\ e_{r\phi}^5 = \frac{1}{4}\, e^r,\ e_{r\phi}^7 = -\frac{1}{4}\, e^r,\ e_{t\phi}^4 = \frac{1}{8}\, e^{2r},\ e_{t\phi}^8 = -\frac{1}{8}\, e^{2r},\\
&e_{rr}^6 = \frac{1}{2},\ e_{tt}^4 = \frac{1}{8}\, e^{2r},\ e_{tt}^8 = \frac{1}{8}\, e^{2r},\ e_{tt}^6 = \frac{1}{4}\, e^{2r},\\
&e_{\phi\phi}^4 = \frac{1}{8}\, e^{2r},\ e_{\phi\phi}^8 = \frac{1}{8}\, e^{2r},\ e_{\phi\phi}^6 = -\frac{1}{4}\, e^{2r}.
\end{aligned}
\tag{B.6}
$$

Then the inverse vielbein exists. An explicit calculation shows that

$$
\begin{aligned}
&E_2^r = 1,\ E_1^t = e^{-r},\ E_3^t = e^{-r},\ E_1^\phi = e^{-r},\ E_3^\phi = -e^{-r},\\
&E_4^{tt} = 4\, e^{-2r},\ E_4^{\phi\phi} = 4\, e^{-2r},\ E_4^{t\phi} = 4\, e^{-2r},\ E_5^{rt} = 2\, e^{-r},\ E_5^{r\phi} = 2\, e^{-r}, E_6^{rr} = \frac{8}{3},\ E_6^{tt} = \frac{4}{3}\, e^{-2r},\\
&E_6^{\phi\phi} = -\frac{4}{3}\, e^{-2r}, E_7^{rt} = 2\, e^{-r},\ E_7^{r\phi} = -2\, e^{-r},\ E_8^{tt} = 4\, e^{-2r},\ E_8^{\phi\phi} = 4\, e^{-2r},\ E_8^{t\phi} = -4\, e^{-2r}
\end{aligned}
\tag{B.7}
$$

Therefore, the 8D local frame spanned by e_μ^a and $e_{\mu\nu}^a$ is actually non-degenerate.

The spin-3 gauge field vanishes.

$$
\phi_{\mu\nu\lambda} = g_{(\mu\nu)\lambda} = 0 \tag{B.8}
$$

$M_{(\mu\nu)(\lambda\rho)} = g_{(\mu\nu)(\lambda\rho)}$ (3.23) have the following non-vanishing components.

$$
\begin{aligned}
g_{(rr)(rr)} &= \frac{1}{3},\ \ g_{(rt)(rt)} = \frac{-1}{4}\, e^{2r}, g_{(r\phi)(r\phi)} = \frac{1}{4}\, e^{2r},\ \ g_{(tt)(tt)} = \frac{1}{3}\, e^{4r},\ \ g_{(t\phi)(t\phi)} = \frac{-1}{4}\, e^{4r},\\
g_{(\phi\phi)(\phi\phi)} &= \frac{1}{3}\, e^{4r},\ \ g_{(rr)(tt)} = \frac{1}{6}\, e^{2r},\ \ g_{(rr)(\phi\phi)} = \frac{-1}{6}\, e^{2r},\ \ g_{(tt)(\phi\phi)} = \frac{1}{6}\, e^{4r}
\end{aligned}
\tag{B.9}
$$

So the tensor $J^{(\mu\nu)(\lambda\rho)}$ (3.24) is given by

$$
\begin{aligned}
J^{(rr)(rr)} &= \frac{16}{3},\ \ J^{(rt)(rt)} = -4\, e^{-2r},\ \ J^{(r\phi)(r\phi)} = 4\, e^{-2r},\ J^{(t\phi)(t\phi)} = -4\, e^{-4r},\\
J^{(tt)(tt)} &= \frac{16}{3}\, e^{-4r}, J^{(\phi\phi)(\phi\phi)} = \frac{16}{3}\, e^{-4r},\ \ J^{(rr)(tt)} = \frac{8}{3}\, e^{-2r},\ \ J^{(rr)(\phi\phi)} = \frac{8}{3}\, e^{-2r},\\
J^{(tt)(\phi\phi)} &= \frac{8}{3}\, e^{-4r}.
\end{aligned}
\tag{B.10}
$$

The killing vectors determine the generalized diffeomorphisms which do not change the metric-like quantities, $g_{\mu\nu}$, $g_{\mu(\nu\lambda)}$ and $g_{(\mu\nu)(\lambda\rho)}$. In the spin-3 geometry there exist two types: Killing vectors ξ_μ and Killing tensors $\xi_{(\mu\nu)}$. They are determined by eqs $\delta g_{\mu\nu} = 0$ and $\delta\phi_{\mu\nu\lambda} = 0$. By the results (5.5) and (5.15), they are determined by the following set of eqs.

$$
\nabla_\mu\, \xi_\nu + \nabla_\nu\, \xi_\mu = 0, \tag{B.11}
$$

$$
\begin{aligned}
&\nabla_\mu\, \left(\xi_{(\nu\lambda)} + \frac{1}{3}\, \rho^a\, \Lambda_a\, g_{\nu\lambda}\right) + \nabla_\nu\, \left(\xi_{(\lambda\mu)} + \frac{1}{3}\, \rho^a\, \Lambda_a\, g_{\lambda\mu}\right) + \nabla_\lambda\, \left(\xi_{(\mu\nu)}\right.\\
&\left.+\frac{1}{3}\, \rho^a\, \Lambda_a\, g_{\mu\nu}\right) = 0
\end{aligned}
\tag{B.12}
$$

Generally, these are coupled eqs for $\delta g_{\mu\nu} = 0$ and $\delta\phi_{\mu\nu\lambda} = 0$. However, if the background geometry is AdS$_3$, $\delta\phi_{\mu\nu\lambda}$ and ρ^a vanish, and the eqs are decoupled. Then ξ_μ and $\xi_{(\mu\nu)}$ are determined by

$$\begin{aligned}
\hat{\nabla}_\mu\,\xi_\nu + \hat{\nabla}_\nu\,\xi_\mu &= 0, && \text{(B.13)}\\
\hat{\nabla}_\mu\,\xi_{(\nu\lambda)} + \hat{\nabla}_\nu\,\xi_{(\lambda\mu)} + \hat{\nabla}_\lambda\,\xi_{(\mu\nu)} &= 0 && \text{(B.14)}
\end{aligned}$$

Here $\hat{\nabla}_\mu$ is the Christoffel symbol for AdS$_3$ background. The Killing tensors *cannot* be expressed in terms of the Killing vectors as $\xi_{(\mu\nu)} = \xi_\mu\,\xi_\nu - \frac{1}{3}\,g_{\mu\nu}\,g^{\lambda\rho}\,\xi_\lambda\,\xi_\rho$.

In this case of AdS$_3$ geometry, there exist 6 Killing vectors $\xi^{(i)}_\mu$ ($i = 1, \ldots, 6$) and 10 Killing tensors $\xi^{(\alpha)}_{(\mu\nu)}$ ($\alpha = 1, \ldots, 10$). The Killing vectors correspond to the isometry $SO(2,2)$, and are the same as those in the spin-2 gravity: $\xi^{(i)} \equiv \xi^{(i)}_\mu\,g^{\mu\nu}\,\partial_\nu$

$$\begin{aligned}
&\xi^{(1)} = \partial_t, \qquad \xi^{(2)} = \partial_\phi, \qquad \xi^{(3)} = \phi\,\partial_t - t\,\partial_\phi\\
&\xi^{(4)} = \partial_r - t\,\partial_t - \phi\,\partial_\phi, \qquad \xi^{(5)} = t\,\partial_r - \frac{1}{2}\,(t^2+\phi^2)\,\partial_t - \frac{1}{2}\,e^{-2r}\,\partial_t - t\,\phi\,\partial_\phi,\\
&\xi^{(6)} = \phi\,\partial_r - t\,\phi\,\partial_t - \frac{1}{2}\,(t^2+\phi^2)\,\partial_\phi - \frac{1}{2}\,e^{-2r}\,\partial_\phi
\end{aligned} \tag{B.15}$$

The Killing tensors are given by

$$\begin{aligned}
&\xi^{(1)}_{(rr)} = 1, \quad \xi^{(1)}_{(tt)} = \frac{3}{2}\,(t^2+\phi^2)\,e^{4r} + \frac{1}{2}\,e^{2r}, \quad \xi^{(1)}_{(\phi\phi)} = \frac{3}{2}\,(t^2+\phi^2)\,e^{4r} - \frac{1}{2}\,e^{2r},\\
&\xi^{(1)}_{(rt)} = \frac{3}{2}\,t\,e^{2r}, \quad \xi^{(1)}_{(r\phi)} = -\frac{3}{2}\,\phi\,e^{2r}, \quad \xi^{(1)}_{(t\phi)} = -3\,t\,\phi\,e^{4r}
\end{aligned}$$

$$\begin{aligned}
&\xi^{(2)}_{(rr)} = t,\ \xi^{(2)}_{(tt)} = \frac{1}{2}\,t\,(t^2+3\,\phi^2)\,e^{4r} + \frac{1}{2}\,t\,e^{2r},\ \xi^{(2)}_{(\phi\phi)} = \frac{1}{2}\,t\,(t^2+3\,\phi^2)\,e^{4r} - \frac{1}{2}\,t\,e^{2r},\\
&\xi^{(2)}_{(rt)} = \frac{3}{4}\,t\,(t^2+\phi^2)\,e^{2r} + \frac{1}{4},\ \xi^{(2)}_{(r\phi)} = -\frac{3}{2}\,t\,\phi\,e^{2r},\ \xi^{(2)}_{(t\phi)} = -\frac{1}{2}\,\phi\,(3\,t^2+\phi^2)\,e^{4r} - \frac{3}{4}\,\phi\,e^{2r}
\end{aligned}$$

$$\begin{aligned}
&\xi^{(3)}_{(rr)} = \phi,\ \xi^{(3)}_{(tt)} = \frac{1}{2}\,\phi\,(3\,t^2+\phi^2)\,e^{4r} + \frac{1}{2}\,\phi\,e^{2r},\ \xi^{(3)}_{(\phi\phi)} = \frac{1}{2}\,\phi\,(3\,t^2+\phi^2)\,e^{4r} - \frac{1}{2}\,\phi\,e^{2r},\\
&\xi^{(3)}_{(rt)} = \frac{3}{2}\,t\phi\,e^{2r},\ \xi^{(3)}_{(r\phi)} = -\frac{3}{4}\,(t^2+\phi^2)\,e^{2r} + \frac{1}{4},\ \xi^{(3)}_{(t\phi)} = -\frac{1}{2}\,t\,(t^2+3\,\phi^2)\,e^{4r}
\end{aligned}$$

$$\begin{aligned}
&\xi^{(4)}_{(rr)} = t\,\phi,\ \xi^{(4)}_{(tt)} = \frac{1}{2}\,t\phi\,(t^2+\phi^2)\,e^{4r} + \frac{1}{2}\,t\,\phi\,e^{2r},\ \xi^{(4)}_{(\phi\phi)} = \frac{1}{2}\,t\phi\,(t^2+\phi^2)\,e^{4r} - \frac{1}{2}\,t\phi\,e^{2r},\\
&\xi^{(4)}_{(rt)} = \frac{1}{4}\,\phi\,(\phi^2+3\,t^2)\,e^{2r} + \frac{1}{4}\,\phi,\ \xi^{(4)}_{(r\phi)} = -\frac{1}{4}\,t(t^2+3\,\phi^2)\,e^{2r} + \frac{1}{4}\,t,\\
&\xi^{(4)}_{(t\phi)} = -\frac{1}{8}\,(t^4+\phi^4+6\,t^2\phi^2)\,e^{4r} + \frac{1}{8}
\end{aligned}$$

$$\begin{aligned}
&\xi^{(5)}_{(rr)} = t^2+\phi^2, \quad \xi^{(5)}_{(tt)} = \frac{1}{4}\,(t^4+6\,t^2\,\phi^2+\phi^4)\,e^{4r} + \frac{1}{2}\,(t^2+\phi^2)\,e^{2r} + \frac{1}{4},\\
&\xi^{(5)}_{(\phi\phi)} = \frac{1}{4}\,(t^4+6\,t^2\,\phi^2+\phi^4)\,e^{4r} - \frac{1}{2}\,(t^2+\phi^2)\,e^{2r} + \frac{1}{4},\\
&\xi^{(5)}_{(rt)} = \frac{1}{2}\,t\,(3\,\phi^2+t^2)\,e^{2r} + \frac{1}{2}\,t, \quad \xi^{(5)}_{(r\phi)} = -\frac{1}{2}\,\phi\,(3\,t^2+\phi^2)\,e^{2r} + \frac{1}{2}\,\phi, \quad \xi^{(5)}_{(t\phi)} = -t\,\phi(t^2+\phi^2)\,e^{4r}
\end{aligned}$$

$$\xi^{(6)}_{(tt)} = e^{4r}, \qquad \xi^{(6)}_{(\phi\phi)} = e^{4r}$$

$$\xi^{(7)}_{(t\phi)} = e^{4r},$$

$$\xi^{(8)}_{(tt)} = \xi^{(8)}_{(\phi\phi)} = 2t\phi\, e^{4r}, \qquad \xi^{(8)}_{(rt)} = \phi\, e^{2r}, \qquad \xi^{(8)}_{(r\phi)} = -t\, e^{2r}, \qquad \xi^{(8)}_{(t\phi)} = -(t^2+\phi^2)\, e^{4r}$$

$$\xi^{(9)}_{(tt)} = 2t\phi\, e^{4r}, \qquad \xi^{(9)}_{(\phi\phi)} = 2t\, e^{4r}, \qquad \xi^{(9)}_{(rt)} = e^{2r}, \qquad \xi^{(9)}_{(t\phi)} = -2\,\phi\, e^{4r}$$

$$\xi^{(10)}_{(tt)} = -2\,\phi\, e^{4r}, \qquad \xi^{(10)}_{(r\phi)} = e^{2r}, \qquad \xi^{(10)}_{(t\phi)} = 2t\, e^{4r}, \qquad \xi^{(10)}_{(\phi\phi)} = -2\,\phi\, e^{4r} \tag{B.16}$$

Those components which are not presented vanish.

The Killing vectors (tensors) are related to the $SL(3,R)$ matrices $\Lambda_a\, t^a$, which generate the generalized diffeomorphisms, by $\xi_\mu = \Lambda_a\, e^a_\mu$ and $\xi_{(\mu\nu)} = \Lambda_a\, e^a_{\mu\nu}$. The Killing vectors can also be obtained by solving eqs

$$\delta\, e^a_\mu = \partial_\mu\, \Lambda^a + f^a{}_{bc}\, \omega^b_\mu\, \Lambda^c = f^a{}_{bc}\, \Sigma^b\, e^c_\mu. \tag{B.17}$$

Here Σ^b are some functions to be determined by Λ^a.

C Metric-like fields

In this appendix various metric-like fields are defined.

Let us recall that a product of two generators of $sl(3,R)$, t_a and t_b, can be reduced to terms which are linear in t_c or proportional to an identity matrix by using the structure constants.

$$t_a\, t_b = \frac{1}{2}\,[t_a,\, t_b] + \frac{1}{2}\,\{t_a,\, t_b\} = \frac{1}{2}\, f_{ab}{}^c\, t_c + \frac{1}{2}\,(d_{ab}{}^c\, t_c + d_{ab}{}^0\, \boldsymbol{I}) \tag{C.1}$$

Therefore all invariants of the local frame transformations can be constructed by contracting f_{abc}, d_{abc} and h_{ab} with e^a_μ and $e^a_{\mu\nu}$.

Now it is easy to expand t_a in terms of e_μ and $e_{\mu\nu}$.

$$t_a = t_b\, \delta^b_a = t_b\,(e^b_\mu\, E^\mu_a + \frac{1}{2}\, e^b_{\mu\nu}\, E^{\mu\nu}_a) = e_\mu\, E^\mu_a + \frac{1}{2}\, e_{\mu\nu}\, E^{\mu\nu}_a \tag{C.2}$$

By using this eq, then the vielbein e^a_μ, $e^a_{\mu\nu}$ can be expressed in terms of E's.

$$e^a_\mu = \frac{1}{2}\,\mathrm{tr}\, t^a\, e_\mu = \frac{1}{2}\,\mathrm{tr}(e_\nu\, E^{a\nu} + \frac{1}{2}\, e_{\nu\lambda}\, E^{a\nu\lambda})\, e_\mu = g_{\mu\nu}\, E^{a\nu} + \frac{1}{2}\, g_{\mu(\nu\lambda)}\, E^{a\nu\lambda}, \tag{C.3}$$

$$\begin{aligned} e^a_{\mu\nu} &= \frac{1}{2}\,\mathrm{tr}\, t^a\, e_{\mu\nu} = \frac{1}{2}\,\mathrm{tr}(e_\lambda\, E^{a\lambda} + \frac{1}{2}\, e_{\lambda\rho}\, E^{a\lambda\rho})\, e_{\mu\nu} \\ &= g_{(\mu\nu)\lambda}\, E^{a\lambda} + \frac{1}{2}\, g_{(\mu\nu)(\lambda\rho)}\, E^{a\lambda\rho} \end{aligned} \tag{C.4}$$

Now, by using this formula (C.2) h_{ab}, d_{abc} and f_{abc} are expressed in terms of E, e and gauge fields. For h_{ab} one obtains

$$\begin{aligned} h_{ab} &= \frac{1}{2}\,\mathrm{tr}\,t_a\,t_b = \frac{1}{2}\,\mathrm{tr}\left(e_\mu\,E_a^\mu + \frac{1}{2}\,e_{\mu\nu}\,E_a^{\mu\nu}\right)\left(e_\lambda\,E_b^\lambda + \frac{1}{2}\,e_{\lambda\rho}\,E_b^{\lambda\rho}\right) \\ &= g_{\mu\nu}\,E_a^\mu\,E_b^\nu + \frac{1}{2}\,g_{(\mu\nu)\lambda}\,(E_a^{\mu\nu}\,E_b^\lambda + E_b^{\mu\nu}\,E_a^\lambda) + \frac{1}{4}\,g_{(\mu\nu)(\lambda\rho)}E_a^{\mu\nu}\,E_b^{\lambda\rho} \end{aligned} \tag{C.5}$$

For d_{abc} one obtains

$$\begin{aligned} d_{abc} &= \frac{1}{2}\,\mathrm{tr}\,\{t_a,\,t_b\}\,t_c \\ &= 2\,E_a^\mu\,E_b^\nu\,E_c^\lambda\,\phi_{\mu\nu\lambda} + E_a^\mu\,E_b^\nu\,E_c^{\lambda\rho}\left(g_{(\mu\nu)(\lambda\rho)} + \frac{1}{6}\,g_{\mu\nu}\mathrm{tr}\,\rho\,e_{\lambda\rho}\right) \\ &\quad + E_a^\mu\,E_b^{\nu\rho}\,E_c^\lambda\left(g_{(\mu\lambda)(\nu\rho)} + \frac{1}{6}\,g_{\mu\lambda}\mathrm{tr}\,\rho\,e_{\nu\rho}\right) + E_a^{\mu\rho}\,E_b^\nu\,E_c^\lambda\left(g_{(\mu\rho)(\nu\lambda)} + \frac{1}{6}\,g_{\nu\lambda}\mathrm{tr}\,\rho\,e_{\mu\rho}\right) \\ &\quad + \frac{1}{4}\,E_a^\mu\,E_b^{\nu\sigma}\,E_c^{\lambda\rho}\,g_{\mu(\nu\sigma)(\lambda\rho)} + \frac{1}{4}\,E_a^{\mu\sigma}\,E_b^\nu\,E_c^{\lambda\rho}\,g_{\nu(\mu\sigma)(\lambda\rho)} + \frac{1}{4}\,E_a^{\mu\rho}\,E_b^{\nu\sigma}\,E_c^\lambda\,g_{\lambda(\nu\sigma)(\mu\rho)} \\ &\quad + \frac{1}{8}\,E_a^{(\mu\rho)}\,E_b^{\nu\sigma}\,E_c^{\lambda\kappa}\,g_{(\mu\rho)(\nu\sigma)(\lambda\kappa)} \end{aligned} \tag{C.6}$$

Here the following manipulation is used.

$$\mathrm{tr}\,\{e_\mu,\,e_\nu\}\,e_{\lambda\rho} = \mathrm{tr}\,(2\,e_{\mu\nu} + \frac{4}{3}\,g_{\mu\nu}\,\boldsymbol{I} + \frac{2}{3}\,g_{\mu\nu}\,\rho)\,e_{\lambda\rho} = 4\,g_{(\mu\nu)(\lambda\rho)} + \frac{2}{3}\,g_{\mu\nu}\,\mathrm{tr}\,\rho\,e_{\lambda\rho} \tag{C.7}$$

Extra gauge fields $g_{(\mu\nu)(\lambda\rho)}$ .... are defined as follows.

$$g_{(\mu\nu)(\lambda\rho)} = \frac{1}{2}\,\mathrm{tr}\,e_{\mu\nu}\,e_{\lambda\rho}, \tag{C.8}$$

$$g_{(\mu\nu)(\lambda\rho)\sigma} = \frac{1}{2}\,\mathrm{tr}\,\{e_{\mu\nu},\,e_{\lambda\rho}\}\,e_\sigma, \tag{C.9}$$

$$g_{(\mu\nu)(\lambda\rho)(\sigma\kappa)} = \frac{1}{2}\,\mathrm{tr}\,\{e_{\mu\nu},\,e_{\lambda\rho}\}\,e_{\sigma\kappa} \tag{C.10}$$

So for the spin-3 gravity gauge fields with up to 6 indices must be introduced. Note that one can also define a gauge field such as

$$\begin{aligned} \phi_{(\mu\nu)(\lambda\rho)} &= \frac{1}{2}\,\mathrm{tr}\,\hat{e}_{\mu\nu}\,\hat{e}_{\lambda\rho} \\ &= g_{(\mu\nu)(\lambda\rho)} + \frac{1}{6}\,g_{\mu\nu}\,\mathrm{tr}\,\rho\,e_{\lambda\rho} + \frac{1}{6}\,g_{\lambda\rho}\,\mathrm{tr}\,\rho\,e_{\mu\nu} + \frac{1}{18}\,g_{\mu\nu}\,g_{\lambda\rho}\,\mathrm{tr}\,\rho^2. \end{aligned} \tag{C.11}$$

Similarly, for f_{abc} one has

$$\begin{aligned} f_{abc} &= \frac{1}{2}\,\mathrm{tr}\,[t_a,\,t_b]\,t_c \\ &= \frac{1}{2}\,E_a^\mu\,E_b^\lambda\,E_c^\sigma\,\mathrm{tr}\,[e_\mu,\,e_\lambda]\,e_\sigma \\ &\quad + \frac{1}{4}\,E_a^\mu\,E_b^\lambda\,E_c^{\sigma\kappa}\,\mathrm{tr}\,[e_\mu,\,e_\lambda]\,e_{\sigma\kappa} + \text{ permutations} \\ &\quad + \frac{1}{8}\,E_a^\mu\,E_b^{\lambda\rho}\,E_c^{\sigma\kappa}\,\mathrm{tr}\,[e_\mu,\,e_{\lambda\rho}]\,e_{\sigma\kappa} + \text{ permutations} \\ &\quad + \frac{1}{16}\,E_a^{\mu\nu}\,E_b^{\lambda\rho}\,E_c^{\sigma\kappa}\,\mathrm{tr}\,[e_{\mu\nu},\,e_{\lambda\rho}]\,e_{\sigma\kappa} \end{aligned} \tag{C.12}$$

Here some terms which can be obtained by permutation of indices are not written explicitly.

Therefore for spin-3 gravity, partly anti-symmetric gauge fields with up to 6 indices such as

$$F_{\mu\lambda\sigma} \equiv \frac{1}{2}\,\mathrm{tr}\,[e_\mu,\, e_\lambda]\, e_\sigma, \qquad F_{\mu\lambda(\sigma\kappa)} \equiv \frac{1}{2}\,\mathrm{tr}\,[e_\mu,\, e_\lambda]\, e_{\sigma\kappa}, \quad \ldots \tag{C.13}$$

must also be introduced. To remove the local frame indices $a, b, ..$ from the action integral and the various relations obtained in this paper it is necessary to use the gauge fields defined in this appendix. However, if these gauge fields are used, the expression will become more complicated. Hence for simplicity, elimination of ${d^a}_{bc}$ and ${f^a}_{bc}$ is not attempted in this paper.

D Solution for $S_{\mu\nu,\lambda\rho}$

In this appendix a solution to the eqs for $S_{\mu\nu,\lambda\rho}$, (3.30), (3.32) are presented. Let us define matrices $\mathcal{A}_{\mu\nu}$ and $\mathcal{B}^{\mu\nu}$ by

$$\mathcal{B}^{\mu\nu} = \frac{5}{48}\, J^{(\mu\nu)(\lambda\rho)}\, W_{\lambda\rho}, \tag{D.1}$$

$$\mathcal{A}_{\mu\nu} = \hat{\nabla}_\mu\, {\phi_{\nu\lambda}}^\lambda - \frac{5}{9}\,\hat{\nabla}^\lambda\, \Phi_{\mu\nu\lambda} - \frac{5}{72}\,(\hat{\nabla}_\mu\, \Phi_{\nu\alpha\beta} + \hat{\nabla}_\nu\, \Phi_{\mu\alpha\beta} - \hat{\nabla}_\alpha\, \Phi_{\mu\nu\beta})\, J^{(\alpha\beta)(\sigma\kappa)}\, W_{\sigma\kappa} \tag{D.2}$$

The matrix $\mathcal{B}^{\mu\nu}$ is symmetric traceless but $\mathcal{A}_{\mu\nu}$ is not symmetric. In terms of these matrices, eq (3.32) is written as

$$g^{\lambda\rho}\, S_{\mu\lambda,\rho\nu} + S_{\mu\nu,\lambda\rho}\, \mathcal{B}^{\lambda\rho} = \mathcal{A}_{\mu\nu}. \tag{D.3}$$

If $\phi_{\mu\nu\lambda} = 0$, then the connections reduce to the Christoffel symbol in Einstein gravity. In this case $\mathcal{A}_{\mu\nu} = 0$ and the solution will be given by $S_{\mu\nu,\lambda\rho} = 0$. In what follows the above matrices are considered to be small and $S_{\mu\nu,\lambda\rho}$ will be obtained as a power series in these matrices. Then the second term on the lefthand side of (D.3) is second order in $\mathcal{A}$, $\mathcal{B}$.

Let us first consider the equation.

$$g^{\lambda\rho}\, S^{(0)}_{\mu\lambda,\rho\nu} = \mathcal{A}_{\mu\nu} \tag{D.4}$$

The solution to this eq and (3.30) is obtained as

$$\begin{aligned} S^{(0)}_{\mu\nu,\lambda\rho} = & -\frac{3}{5}\, g_{\mu\nu}\,(\mathcal{A}_{\lambda\rho} + \mathcal{A}_{\rho\lambda}) - \frac{2}{5}\, g_{\lambda\rho}\,(\mathcal{A}_{\mu\nu} + \mathcal{A}_{\nu\mu}) \\ & -\frac{3}{10}\,(g_{\mu\lambda}\, g_{\nu\rho} + g_{\mu\rho}\, g_{\nu\lambda} - 2\, g_{\mu\nu}\, g_{\lambda\rho})\, {\mathcal{A}_\alpha}^\alpha \\ & +\frac{1}{5}\, g_{\mu\lambda}\,(2\,\mathcal{A}_{\nu\rho} + \mathcal{A}_{\rho\nu}) + \frac{1}{5}\, g_{\nu\lambda}\,(2\,\mathcal{A}_{\mu\rho} + \mathcal{A}_{\rho\mu}) \\ & +\frac{1}{5}\, g_{\mu\rho}\,(2\,\mathcal{A}_{\nu\lambda} + \mathcal{A}_{\lambda\nu}) + \frac{1}{5}\, g_{\nu\rho}\,(2\,\mathcal{A}_{\mu\lambda} + \mathcal{A}_{\lambda\mu}) \end{aligned} \tag{D.5}$$

Now we split $S_{\mu\nu,\lambda\rho}$ as $S_{\mu\nu,\lambda\rho} = S^{(0)}_{\mu\nu,\lambda\rho} + \tilde{S}_{\mu\nu,\lambda\rho}$. Owing to (D.4) this remaining part $\tilde{S}_{\mu\nu,\lambda\rho}$ satisfies

$$g^{\lambda\rho}\,\tilde{S}_{\mu\lambda,\rho\nu} + \tilde{S}_{\mu\nu,\lambda\rho}\,\mathcal{B}^{\lambda\rho} = \mathcal{A}^{(1)}_{\mu\nu}, \tag{D.6}$$

where $\mathcal{A}^{(1)}_{\mu\nu} = -S^{(0)}_{\mu\nu,\lambda\rho}\,\mathcal{B}^{\lambda\rho}$. This equation has the same structure as (D.3). Because the term $\tilde{S}_{\mu\nu,\lambda\rho}\,\mathcal{B}^{\lambda\rho}$ is subleading, we can drop this term to leading order, replace $g^{\lambda\rho}\,\tilde{S}_{\mu\lambda,\rho\nu}$ by $g^{\lambda\rho}\,S^{(1)}_{\mu\lambda,\rho\nu}$, and solve eq $g^{\lambda\rho}\,S^{(1)}_{\mu\lambda,\rho\nu} = \mathcal{A}^{(1)}_{\mu\nu}$, which has the same form as (D.4).

We will repeat this procedure, obtain $S^{(n)}_{\mu\nu,\lambda\rho}$ at each step and by assuming convergence finally sum up $S^{(n)}_{\mu\nu,\lambda\rho}$ to have $S_{\mu\nu,\lambda\rho} = \sum_{n=0}^{\infty} S^{(n)}_{\mu\nu,\lambda\rho}$. $\mathcal{A}^{(n)}_{\mu\nu}$ is defined by

$$\mathcal{A}^{(n)}_{\mu\nu} = -S^{(n-1)}_{\mu\nu,\lambda\rho}\,\mathcal{B}^{\lambda\rho} \qquad (\mathcal{A}^{(0)}_{\mu\nu} = \mathcal{A}_{\mu\nu}). \tag{D.7}$$

The equation for $S^{(n)}_{\mu\nu,\lambda\rho}$ is given by

$$g^{\lambda\rho}\,S^{(n)}_{\mu\lambda,\rho\nu} = \mathcal{A}^{(n)}_{\mu\nu}. \tag{D.8}$$

and the solution is given by (D.5) with $S^{(0)}_{\mu\nu,\lambda\rho}$ and $\mathcal{A}_{\mu\nu}$ replaced by $S^{(n)}_{\mu\nu,\lambda\rho}$ and $\mathcal{A}^{(n)}_{\mu\nu}$, respectively.

The above procedure yields recursion relations between $\mathcal{A}^{(n)}_{\mu\nu}$ and $\mathcal{A}^{(n+1)}_{\mu\nu}$, and their solution for $\mathcal{A}^{(n)}_{\mu\nu}$ takes the following form.

$$\begin{aligned}\mathcal{A}^{(n)}_{\mu\nu} \quad &= \quad g_{\mu\nu}\,F^{(n)} + \sum_{m=1}^{n-1}(\mathcal{B}^m)_{\mu\nu}\,X^{(n)}_m \\ &\quad + \sum_{m=0}^{n}\{(\mathcal{B}^m\,\mathcal{A}\,\mathcal{B}^{n-m})_{\mu\nu} + (\mathcal{B}^m\,\mathcal{A}\,\mathcal{B}^{n-m})_{\nu\mu}\}\,Y^{(n)}_m\end{aligned} \tag{D.9}$$

Here $g_{\mu\nu}$ is inserted to construct the powers of $\mathcal{B}^{\lambda\rho}$. By computing $S^{(n)}_{\mu\nu,\lambda\rho}$ by using (D.5) with suitable replacements and obtaining $\mathcal{A}^{(n+1)}_{\mu\nu}$ by using (D.7), the recursion relations for $F^{(n)}$, $X^{(n)}_m$ and $Y^{(n)}_m$ are obtained.

$$F^{(n+1)} \quad = \quad \frac{6}{5}\sum_{m=1}^{n-1}\mathrm{tr}(\mathcal{B}^{m+1})\,X^{(n)}_m + \frac{12}{5}\sum_{m=0}^{n}Y^{(n)}_m\,\mathrm{tr}(\mathcal{A}\,\mathcal{B}^{n+1}), \tag{D.10}$$

$$X^{(n+1)}_1 \quad = \quad -\frac{3}{5}\,F^{(n)} + \frac{6}{5}\,\mathrm{tr}(\mathcal{A}\mathcal{B}^n)\sum_{m=0}^{n}Y^{(n)}_m + \frac{3}{5}\sum_{m=1}^{n-1}(\mathrm{tr}\,\mathcal{B}^m)\,X^{(n)}_m, \tag{D.11}$$

$$X^{(n+1)}_{m+1} \quad = \quad -\frac{12}{5}\,X^{(n)}_m, \qquad Y^{(n+1)}_m = -\frac{6}{5}\,Y^{(n)}_m - \frac{6}{5}\,Y^{(n)}_{m-1} \tag{D.12}$$

The initial condition at $n = 1$ is

$$Y^{(1)}_0 = -\frac{4}{5}, \qquad Y^{(1)}_1 = -\frac{2}{5}, \qquad F^{(1)} = \frac{6}{5}\,\mathrm{tr}(\mathcal{B}\mathcal{A}), \qquad X^{(1)}_1 = \frac{3}{5}\,\mathrm{tr}\mathcal{A} \tag{D.13}$$

The solution for $Y^{(n)}_m$ is given by

$$Y^{(n)}_m = \frac{1}{3}\left(-\frac{6}{5}\right)^n\frac{(n-1)!}{m!\,(n-m)!}\,(2n-m). \tag{D.14}$$

$X_m^{(n)}$ is determined in terms of $F^{(m)}$ by $X_m^{(n)} = (3/4)\,(-12/5)^m\, F^{(n-m)}$, and $F^{(n)}$ is the solution to the recursion relation

$$F^{(n+1)} = \frac{6}{5} \sum_{m=1}^{n-1} (-12/5)^m \,\mathrm{tr}\,(\mathcal{B}^{m+1})\, F^{(n-m)} - \frac{1}{2}\,(-12/5)^{n+1}\,\mathrm{tr}(\mathcal{A}\mathcal{B}^{n+1}). \tag{D.15}$$

This last eq can be solved by iterations.

$$\begin{aligned} F^{(n)} &= -\frac{1}{2}\,(-12/5)^n\,\mathrm{tr}(\mathcal{A}\mathcal{B}^n) + \frac{1}{4}\sum_{m=1}^{n-2}(-12/5)^n\,\mathrm{tr}(\mathcal{B}^{n-m})\,\mathrm{tr}(\mathcal{A}\mathcal{B}^m) \\ &\quad +(25/576)\sum_{m=1}^{n-2}\sum_{k=1}^{m-2}(-12/5)^{n-k}\,\mathrm{tr}(\mathcal{B}^{n-m})\,\mathrm{tr}(\mathcal{B}^{m-k})\,F^{(k)} = \ldots \end{aligned} \tag{D.16}$$

For example, $F^{(2)} = -(72/25)\,\mathrm{tr}(\mathcal{A}\mathcal{B}^2)$, $F^{(3)} = -(432/125)\,\mathrm{tr}(\mathcal{B}^2)\,\mathrm{tr}(\mathcal{A}\mathcal{B})+(864/125)\,\mathrm{tr}(\mathcal{A}\mathcal{B}^3)$, $F^{(4)} = (5184/625)\,\mathrm{tr}(\mathcal{B}^2)\,\mathrm{tr}(\mathcal{A}\mathcal{B}^2)+(5184/625)\,\mathrm{tr}(\mathcal{B}^3)\,\mathrm{tr}(\mathcal{A}\mathcal{B})-(10368/625)\,\mathrm{tr}(\mathcal{A}\mathcal{B}^4)$ and then we obtain $X_1^{(2)} = -(54/25)\,\mathrm{tr}(\mathcal{B}\mathcal{A})$, $X_1^{(3)} = (648/125)\,\mathrm{tr}(\mathcal{B}^2\mathcal{A})$, $X_2^{(3)} = (648/125)\,\mathrm{tr}(\mathcal{B}\mathcal{A})$. Then, $\mathcal{A}_{\mu\nu}^{(n)}$ can be computed using (D.9) to any desired larger value of n.

Finally, $S_{\mu\nu,\lambda\rho}$ is given by

$$\begin{aligned} S_{\mu\nu,\lambda\rho} &= \sum_{n=0}^{\infty} S_{\mu\nu,\lambda\rho}^{(n)} \\ &= \sum_{n=0}^{\infty}\Big\{ -\frac{3}{5}\,g_{\mu\nu}\,(\mathcal{A}_{\lambda\rho}^{(n)} + \mathcal{A}_{\rho\lambda}^{(n)}) - \frac{2}{5}\,g_{\lambda\rho}\,(\mathcal{A}_{\mu\nu}^{(n)} + \mathcal{A}_{\nu\mu}^{(n)}) \\ &\quad + \frac{3}{5}\,g_{\mu\nu}\,g_{\lambda\rho}\,\mathcal{A}_{\alpha}^{(n)\alpha} - \frac{3}{10}\,(g_{\mu\lambda}\,g_{\nu\rho} + g_{\mu\rho}\,g_{\nu\lambda})\,\mathcal{A}_{\alpha}^{(n)\alpha} \\ &\quad + \frac{1}{5}\,g_{\mu\lambda}\,(2\,\mathcal{A}_{\nu\rho}^{(n)} + \mathcal{A}_{\rho\nu}^{(n)}) + \frac{1}{5}\,g_{\nu\lambda}\,(2\,\mathcal{A}_{\mu\rho}^{(n)} + \mathcal{A}_{\rho\mu}^{(n)}) \\ &\quad + \frac{1}{5}\,g_{\mu\rho}\,(2\,\mathcal{A}_{\nu\lambda}^{(n)} + \mathcal{A}_{\lambda\nu}^{(n)}) + \frac{1}{5}\,g_{\nu\rho}\,(2\,\mathcal{A}_{\mu\lambda}^{(n)} + \mathcal{A}_{\lambda\mu}^{(n)}) \Big\} \end{aligned} \tag{D.17}$$

E Metric G_{MN} for '8D space'

The metric tensor $g_{\mu\nu}$ is constructed in terms of e_μ^a. By a similar construction one can extend the metric tensor to that for an extended 8D space by combining e_μ^a and $e_{\mu\nu}^a$. Let us define a new metric

$$\begin{pmatrix} G_{\mu\lambda} & G_{\mu(\rho\kappa)} \\ G_{(\nu\sigma)\lambda} & G_{(\nu\sigma)(\rho\kappa)} \end{pmatrix}$$

by the eqs.

$$G_{\mu\nu} = g_{\mu\nu} = e_\mu^a\, e_{a\nu}, \tag{E.1}$$

$$G_{\mu(\nu\lambda)} = G_{(\nu\lambda)\mu} = g_{\mu(\nu\lambda)} = e_\mu^a\, e_{a\nu\lambda}, \tag{E.2}$$

$$G_{(\mu\nu)(\lambda\rho)} = g_{(\mu\nu)(\lambda\rho)} = e_{\mu\nu}^a\, e_{a\lambda\rho} \tag{E.3}$$

This tensor is a metric tensor in a fictitious 8D space which contains the ordinary spacetime. The inverse metric is easily obtained.

$$G^{\mu\nu} = g^{\mu\nu} + \frac{1}{4} g^{\mu}{}_{(\lambda\rho)} J^{(\lambda\rho)(\sigma\kappa)} g_{(\sigma\kappa)}{}^{\nu}, \tag{E.4}$$

$$G^{\mu(\nu\lambda)} = G^{(\nu\lambda)\mu} = -\frac{1}{2} g^{\mu}{}_{(\rho\sigma)} J^{(\rho\sigma)(\nu\lambda)}, \tag{E.5}$$

$$G^{(\mu\nu)(\lambda\rho)} = J^{(\mu\nu)(\lambda\rho)} \tag{E.6}$$

$J^{(\lambda\rho)(\sigma\kappa)}$ is defined in (3.24). They satisfy the following relations.

$$G^{\mu\nu} G_{\nu\lambda} + \frac{1}{2} G^{\mu(\nu\rho)} G_{(\nu\rho)\lambda} = \delta^{\mu}_{\lambda}, \tag{E.7}$$

$$G^{\mu\nu} G_{\nu(\lambda\rho)} + \frac{1}{2} G^{\mu(\nu\sigma)} G_{(\nu\sigma)(\lambda\rho)} = 0, \tag{E.8}$$

$$G^{(\mu\nu)\lambda} G_{\lambda\rho} + \frac{1}{2} G^{(\mu\nu)(\lambda\sigma)} G_{(\lambda\sigma)\rho} = 0, \tag{E.9}$$

$$G^{(\mu\nu)\sigma} G_{\sigma(\lambda\rho)} + \frac{1}{2} G^{(\mu\nu)(\kappa\sigma)} G_{(\kappa\sigma)(\lambda\rho)} = \delta^{\mu}_{\lambda} \delta^{\nu}_{\rho} + \delta^{\nu}_{\lambda} \delta^{\mu}_{\rho} - \frac{2}{3} g^{\mu\nu} g_{\lambda\rho} \tag{E.10}$$

This inverse metric can be expressed in terms of the inverse vielbein. One can show that

$$\begin{aligned} G^{\mu\nu} &= E^{\mu}_{a} E^{a\nu}, \\ G^{\mu(\nu\lambda)} &= E^{\mu}_{a} E^{a\nu\lambda}, \\ G^{(\mu\nu)(\lambda\rho)} &= E^{\mu\nu}_{a} E^{a\lambda\rho}. \end{aligned} \tag{E.11}$$

By using the above formulae one can show that E's can be expanded in terms of e's as follows.

$$E^{a\mu} = G^{\mu\nu} e^{a}_{\nu} + \frac{1}{2} G^{\mu(\nu\lambda)} e^{a}_{\nu\lambda}, \tag{E.12}$$

$$E^{a\mu\nu} = G^{(\mu\nu)\lambda} e^{a}_{\lambda} + \frac{1}{2} G^{(\mu\nu)(\lambda\rho)} e^{a}_{\lambda\rho} \tag{E.13}$$

Let us denote the above metric as G_{MN}, where M and N take two types of indices μ and $(\nu\lambda)$. Then, the relation (E.7)-(E.10) can be succinctly written as

$$G_{MN} G^{NL} = \delta_{M}{}^{L}, \tag{E.14}$$

where $\delta_{M}{}^{L}$ is the ordinary Kronecker's δ symbol for $M = \mu$ and $L = \nu$. Otherwise,

$$\delta_{\mu}{}^{(\nu\lambda)} = \delta_{(\mu\nu)}{}^{\lambda} = 0, \qquad \delta_{(\mu\nu)}{}^{(\lambda\rho)} = \delta^{\lambda}_{\mu} \delta^{\rho}_{\nu} + \delta^{\lambda}_{\nu} \delta^{\rho}_{\mu} - \frac{2}{3} g_{\mu\nu} g^{\lambda\rho}. \tag{E.15}$$

Finally, the covariant derivative ∇_{μ} is compatible with this metric tensor. As was shown in (4.25), $G_{\mu\nu} = g_{\mu\nu}$ satisfies $\nabla_{\lambda} g_{\mu\nu} = 0$. This property is true for all components.

$$\nabla_{\mu} G_{MN} = \partial_{\mu} G_{MN} - \Gamma^{K}_{\mu M} G_{KN} - \Gamma^{K}_{\mu N} G_{KM} = 0. \tag{E.16}$$

This is because G_{MN} is given by $e^a_M\, e_{aN}$ and the vielbeins are covariantly constant, $D_\mu\, e^a_N = 0$. The above relation is not sufficient to determine $\Gamma^N_{\mu M}$ completely, since this is not symmetric under interchange of the lower indices.

When $\nabla_\mu\, G_{(\nu\lambda)(\rho\sigma)}$ is computed explicitly, this does not seemingly vanish. The result contains $g_{(\mu\nu)(\lambda\rho)\sigma}$. As was discussed at the end of sec.2, however, not all the gauge fields are independent. Those relations among these fields will be such that these covariant derivatives of G_{MN} actually vanish. Thus by using the condition of metric-compatibility (E.16), some of these relations may be obtained.

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